PARAMETER ESTIMATION ERROR:
A CAUTIONARY TALE IN COMPUTATIONAL FINANCE

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A CAUTIONARY TALE IN COMPUTATIONAL FINANCE

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I dedicate this thesis to the memory of my father, to whom I owe a great deal — though he would dispute this if he were still alive. Only later in my life was I fortunate to realize the many facets of

V. P. Popovic, D.Sc.

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Mine are just words that acknowledge the harder deeds of others who have helped me along the way. I owe a great debt to several of my professors at this institution. They not only helped me learn probability, statistics, and the usefulness and beauty of simulation, but also became friends during my time at Georgia Tech and so made my life a little richer. In some sort of random order during this excursion, I think of and am grateful for my acquaintance with Robert Kertz, Steve Hackman, and my Committee Chair Dave Goldsman.

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SUMMARY

We quantify the effects on contingent claim valuation of using an estimator for the volatility σ of a geometric Brownian motion (GBM) process. That is, we show what difficulties can arise when failing to account for estimation risk. Our working problem uses a direct estimator of volatility based on the sample standard deviation of increments from the underlying Brownian motion. After replacing the direct estimator into the GBM, we derive the resulting distribution function of the approximated GBM for any time point. This allows us to present post-estimation distributions and valuation formulae for an assortment of European contingent claims that are in accord with many of the basic properties of the underlying risk-neutral process, and yet accurately reflect some of the additional quantifiable uncertainties that exist in a Black–Scholes–Merton type of economy.

Next we extend our work to the contingent claim sensitivities associated with an assortment of European option portfolios that are based on the direct estimator of the volatility σ of the GBM process. Our approach to the option sensitivities — the Greeks — uses the likelihood function technique. This allows us to obtain computable results for the technically more-complicated formulae associated with our post-estimation process. We discuss an assortment of difficulties that can ensue when failing to account for estimation risk in valuation and hedging formulae.

CHAPTER I

INTRODUCTION

In developing theory and in implementing best practices in quantitative finance, there is a distinction between the Black, Scholes, and Merton (BSM) equation and the BSM model. One can use — as traders and speculators do — the BSM equation and its associated Greeks as a convenient "best practice" tool without fully subscribing to the BSM model. We concur with that distinction in what follows, and in the process attempt in this thesis to make the transition between the two views less fragmented. The model assumes, in a well-defined sense, that the *present is a perfect probabilistic reflection of the future*. In other words, this crucial assumption posits that we have perfect knowledge of the parameters instantiating the model. At the level of "best practice," this is certainly not the case.

At the very least, we do not know the volatility σ — a crucial parameter. In fact, we do not exactly know how the volatility propagated in the past, nor do we know the future behavior of σ. The naive model dictates that we ascribe to the notion of *perfect foresight* in the parameter space — exact knowledge of σ. On the other hand, our contribution is a generalization incorporating some form or presumption of learning and adaptation to the environment in which economic agents find themselves — somewhat along the lines of the rational expectations approach. The crux of this point of view was initially introduced by Muth [44] and later developed by Lucas [39, 40]. An implication of the Rational Expectations Hypothesis (REH) is that agents have a vested interest in acquiring and processing information efficiently. According to Lucas [40], "[REH] is a property likely to be (approximately) possessed by the *outcome* of this unspecified process of learning and adapting." An important and extreme implication of the REH is that the objective and subjective cumulative distribution functions (c.d.f.'s) of important economy wide parameters coincide in equilibrium. It follows that economic agents use appropriate inferential techniques, i.e., those geared to the actual processes governing the economy. Arising from the outcomes of agents' optimizing decisions is the testable implication that no systematic errors are generated on account of these very decisions, nor a fortiori from the models that purport to mimic or capture this economic behavior.

In our case, we explicitly incorporate distributional assumptions that economic agents need to deal with in the process of discovering the value of contingent claims. Of the possible set of distributional assumptions, we have chosen a reasonable one — dependent on the statistically well-grounded estimator for the variance of a normal distribution [30] — and capable of accounting for the risk that economic agents operating in the contingent claims markets face. Furthermore, the results that accrue from our "perturbation" of standard BSM are implementable, and we believe can be further generalized to more-sophisticated models along the lines presented in this thesis. Here, in particular, we refer to models that extend themselves to a more-realistic view of volatility creation.

Consider implied volatility — a market construct with numerous hidden assumptions. Best practice only requires that we use — not necessarily even in a consistent way — components of the model, i.e., the BSM equation, to value and hedge our portfolios. In this regard, one of the things we offer in this thesis is a better blend of best practice techniques. We explicitly use the model, paying close attention to its assumptions, and because of our mathematical shortcomings, where necessary, note verbally when we jump from a "model" to a "best practice" approach. Surprisingly, based on the estimator of realized volatility we use, a *single* new parameter "n" — corresponding to the degrees of freedom (df) of a certain chi-square random variable — that embeds itself in all the derived European valuation and hedging formulae that comport with our version of estimation risk.

There are modeling questions concerning n and there are "best practice" questions when dealing with this parameter. At the modeling level we know precisely what is meant by df. The greater the degrees of freedom, the better we approximate the classic BSM model. At the best practice level, we can take n as a hyper-parameter indicating our confidence in the data and to what extent the model assumptions are applicable. Here, for instance, we may model n as a Bayesian — with some prior distribution, or we can take n (not necessarily integer) into account in a direct numeric calibration of our best practice equations. These latter interpretive aspects of n we leave to future work. In what follows we strictly tie n to the df.

Figure 1 depicts four objective c.d.f.'s of an equity's price $S(t;\widehat{\sigma}_n)$ at some future time t, where the equity is instantiated by its current price s and a particular best estimate of volatility $\widehat{\sigma}_n$, depending on the available amount of current information (e.g., $n = 3, 4, 10, 100$). If the BSM model were a perfect depiction of reality, then by the self-similar nature of Brownian motion (BM), over *any* interval of time — assuming sampling is costless — we can get arbitrarily close to the true value of σ by choosing n sufficiently large. In Figure 1, if we choose $n = 1000$, the resulting estimation-dependent c.d.f. is essentially coincident with the c.d.f. of the true GBM governing the stock price $S(t;\sigma)$. Hence, σ is effectively known and we can proceed to valuation using the BSM *model*. Of course, one of several problems is that high-frequency tick data encountered in practice fail to conform to the assumptions underlying GBM. For example, such data may have correlated increments. Another problem is heteroskedasticity, where volatility changes over time. In addition to these difficulties, during periods of financial turmoil, the ratio of implied to realized volatility fluctuates to values substantially different from one.

The typical best practice answer to such difficulties in many large financial institutions is to simply avoid the issues dealing with the uncertainty associated with parameter estimation. Rather, a portfolio is marked-to-market daily and the implied parameters are backed out in some manner and taken more-or-less myopically to represent, as well, their future values. The problem with this approach is that for the volatility specifications that are often used, a one-to-one mapping between the objective and volatility functions does not exist. The usual objective criterion is the minimization of a weighted (typically by the shares of the financial instruments held in the portfolio) sum of least squares (the "squares" components are usually taken as the difference between the actual component portfolio values and the values of a reference set). Given the "backed-out" parameters — with no associated confidence intervals or like measures of belief — a calibration of the portfolio is implemented. Since the backed-out set of parameters are related in some way to the implied volatility structure of the portfolio, one often loses the intuition, due to the lack of

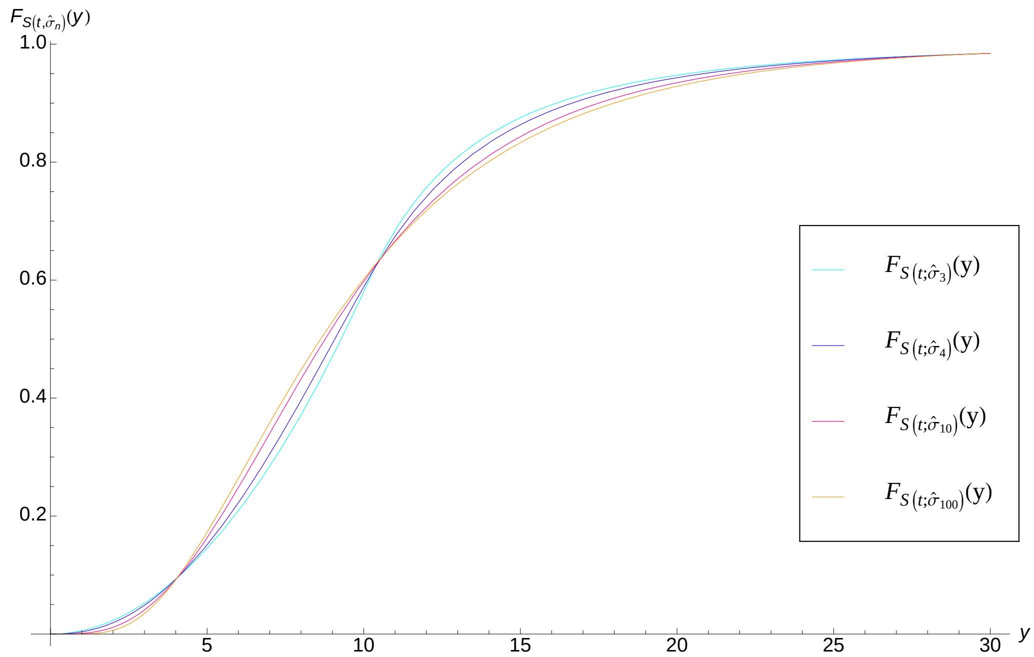


Figure 1: Some possible c.d.f's. of a post-estimation equity price

parameter uniqueness, between the connection of what the volatility actually is and what is reflected from the portfolio and the chosen hedge. Even if one obtains a good least-squares fit for the vanilla portfolio, there still remains the issue of valuing and hedging exotic contracts. As Schoutens et al. [54] point out, it is possible to get near-perfect calibrations on a set of vanilla claims for a wide variety of valuation models, yet miserably fail when pricing path-dependent claims on the basis of the implied parameters. This is a primary reason why the valuation of mortgage-backed securities via the risk-neutral technology has been unsuccessful. In most cases, mortgage bundles are valued by a fitting procedure. This is fine, provided markets are stable, but can result in a costly event when major financial uncertainties reveal themselves. The option markets will reflect substantial market deterioration, but the fitted models reflect little or no change in valuation nor the need for adjustment to a particular hedging strategy.

In a general equilibrium context, for any given set of underlying stocks there is a market for options characterized by an option chain for each named equity. Options are redundant (see the replication argument in Section 3.5.1.1), but due to costs of transacting (explicit and implicit), they are extremely useful in practice. Ideally, at any moment, an option chain on a given underlying specifies the equilibrium prices at which options of given expiries and strikes trade. Unfortunately, it is typical that a chain's record consists of a set of time-asynchronous trading prices. The chain also fails to account for the costs associated with the acquisition of information and the explicit costs associated with transacting at prices away from at-the-money.

Due to the above and because option prices are determined by so-called demand and supply schedules that summarize the technology and preferences of market participants, it is seldom the case that the chain equilibrium reflects the same implied volatility value for each strike and expiry pair. In some sense the option chain's implied volatility structure is similar to that of a multi-factor swaption model [24], where the swaption values are governed by different overlapping random sources. The component prices in a chain are interlinked. A change in the excess demand for a particular strike-expiry repercusses throughout all other components of the chain.

In a BSM world of constant volatility, though the chain's prices will differ, all implied volatilities derived from the chain data are the same. This is not observed in practice. One rationalization for this can be the existence of market participants who are not strictly price takers, each placing different bets on one or another segment of the chain. Their market-expressed beliefs, dealing with the underlying risk and uncertainty, can imply a non-constant implied volatility surface. An example of such a situation is Microsoft — a firm that engages in the option market for its own stock and whose objectives and firm-specific information, at any point in time, are not fully known. Such a firm's market actions can lead to a relatively higher demand for their options on a chain segment, which in turn results in a higher relative option price on that segment, and consequently varying implied volatility over the chain.

The following are several admittedly contrived examples. Their purpose is to indicate what type of estimators would *not* be used by modelers ascribing to the REH. As a modeling question, on rational expectations grounds, both of the example estimators below are incongruent with the processes that the economy is operating under and so should be rejected. Suppose that σ is not a known constant, but rather an observational random variable distributed according to some law. In Figure 2, we have imposed two types of volatility laws. In the top panel, for any a and b, it can be shown that the p.d.f. of the equity price $S(t; a, b)$ at time t under the "uniform law" for volatility is

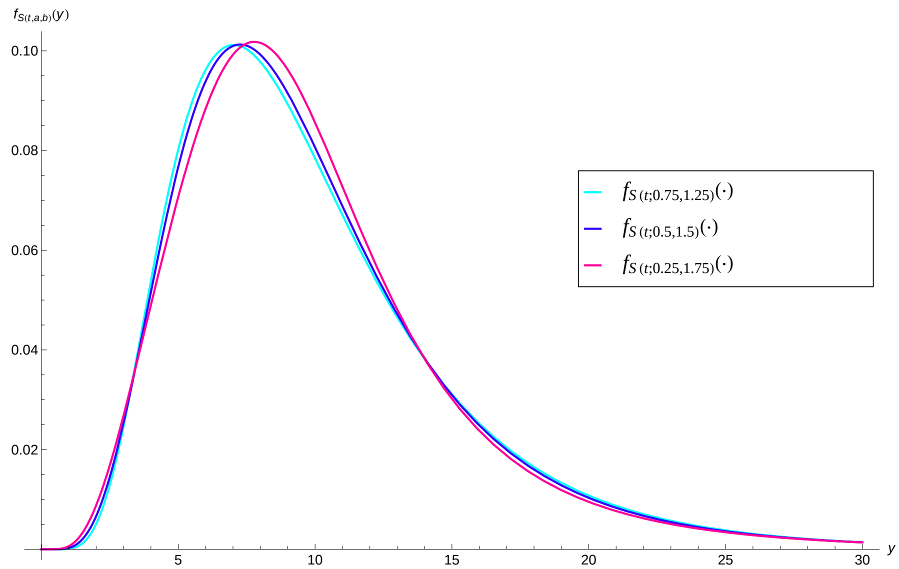
$$f_{S(t;a,b)}(y) \quad = \quad \int_a^b \Phi\left(\frac{\ell\mathrm{n}\left(\frac{y}{s}\right) - \left(\mu - \frac{w}{2}\right)t}{\sqrt{wt}}\right)\frac{1}{b-a}\mathrm{d}w, \quad y > 0.$$

Unlike the volatility estimation law that we use, the above assumes that a and b are known. A similar set-up is used by Buff [14] under a stochastic volatility approach and called the "worst-case" scenario. To be fully comparable to our approach, the appropriate p.d.f. really should be $f_{S(t;\widehat{a}_n,\widehat{b}_n)}(\cdot)$, where a "hat" over a parameter indicates its respective estimator. Suppose that we set $a = 0$. Then, as a reasonable estimator for b, one can choose $\widehat{b}_n = M + \frac{M}{n}$, where n is the sample size and M is a random variable denoting the observed maximum of the sampled volatility draw. In particular, we need to specify the c.d.f. $F_{S(t;\widehat{b}_n)}(y)$. In line with this goal, we require the p.d.f. of $\hat{b}_n$, $f_{\widehat{b}_n}(\cdot)$. Evidently, to this end we can use $\Pr(M \leq \frac{n}{1+n}y)$, but we will not pursue this further here. In the bottom panel of Figure 2 we have depicted an "exponential law" $\mathrm{Exp}(\lambda)$ for volatility, giving rise to the p.d.f.

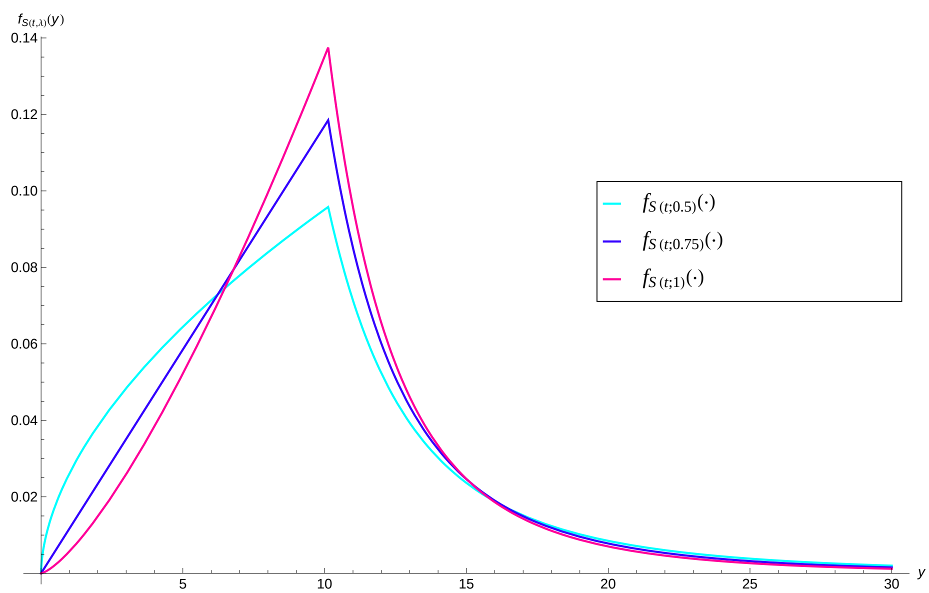
$$f_{S(t;\lambda)}(y) \quad = \quad \int_0^\infty \frac{\mathrm{e}^{-w\lambda - \frac{(\ell\mathrm{n}(\frac{y}{s}) - (\mu - \frac{w}{2})t)^2}{2wt}}\lambda}{\sqrt{2\pi wt}\,y}\mathrm{d}w, \quad y > 0.$$

Once again properties of an estimator for λ fail to be reflected. An estimator for λ is $\widehat{\lambda}_n = 1/\bar{R}$ — which consists of the sampled mean $\bar{R}$ of realized log-returns (see Chapter 2 for a definition of log-returns). But what we really need for this case is the p.d.f. $f_{S(t;\widehat{\lambda}_n)}(\cdot)$ — which reflects the distributional assumptions of the estimator of σ. We note that for any particular instantiation of the parameters for each of the above densities it can be shown that $\mathrm{E}[S(t; \widehat{\sigma}_n)] = s\mathrm{e}^{\mu t}$, which is in accord with the forward price of the underlying when μ equals the risk-free interest rate r.

The moral of our story and the tale we pursue in the chapters that follow, is that estimation risk should not be divorced from the modeling procedure. Rather, risk must be incorporated in the decision process and needs to be based on the underlying structure that the economy presents. We stress the need to incorporate estimation risk — something



(a) Uniform $f_{S(t;a,b)}(\cdot)$ p.d.f's.



(b) Exponential $f_{S(t;\lambda)}(\cdot)$ p.d.f's.

Figure 2: A cornucopia of volatility law p.d.f's.

we can control for — as compared to uncertainty in the sense of Knight [34] — for which we have little to say. Knightian uncertainty in the above examples would be reflected by non-informative prior distributions being placed on a and b in the first example, or λ in the second example. With regard to risk, in this thesis we show, analytically and by example, that it is important to pay attention to all the option Greeks, i.e., option sensitivities, with special emphasis placed on δ (delta), γ (gamma), and ϑ (vega) — discussed in the main chapters of this thesis. Essentially, all our modeling is in line with the REH of Muth and Lucas, in that agents populating the economy or modelers proposing an artificial economy, are required to use inferential techniques that are in accord with the processes that nature has endowed the economy to follow.

The thesis is organized as follows. Chapter §2 derives the direct estimator and shows its applicability to numerous types of European contingent claims. Chapter §3 extends the consequences of the direct estimator to the option sensitivity formulae. This chapter also provides for purposes of comparison multiple examples of portfolio hedges instituted under the standard GBM (pre-estimation) case and our post-estimation case. A referral to and review of the applicable literature is interspersed throughout the two main thesis chapters. Chapter §4 concludes the thesis and suggests applicable areas of future research. Each of the two main chapters has self-contained appendices. The end appendix of the thesis §4.2 contains a sampling of assorted R [50] programs used to construct some of the various figures, tables, and examples interspersed throughout the body of work.

CHAPTER II

ON VALUING AND HEDGING EUROPEAN OPTIONS WHEN VOLATILITY IS ESTIMATED DIRECTLY

In this chapter we quantify the effects on contingent claim valuation of using an estimator for the volatility σ of a geometric Brownian motion (GBM) process. That is, we show what difficulties can arise when failing to account for estimation risk. Our working problem uses a direct estimator of volatility based on the sample standard deviation of increments from the underlying Brownian motion. After replacing the direct estimator into the GBM, we derive the resulting distribution function of the approximated GBM for any time point. This allows us to present post-estimation distributions and valuation formulae for an assortment of European contingent claims that are in accord with many of the basic properties of the underlying risk-neutral process, and yet accurately reflect some of the additional quantifiable uncertainties that exist in a Black–Scholes–Merton type of economy.

2.1 Introduction

The estimation of volatility is a crucial component in understanding the time-series properties of financial markets and the claims they trade. The purpose of the present paper is to provide a better understanding of one key behavior characteristic of hedgers and speculators in the options markets written on some specified equity — namely, how do these market agents cope with occasional *unanticipated* changes in the duration and magnitude of the volatility. We shall assume that the equity price follows a constant-coefficients geometric Brownian motion (GBM) process, but with an unknown volatility parameter. The choice of an equity price as the "underlying" is just a convenience, since when discussing option markets similar results apply to foreign exchange rates, swap rates, and LIBOR rates. Although constant-coefficients GBM does not reflect the exact response of financial variables to an assortment of intrinsic economic forces — for example, *anticipated* transient or permanent jumps in volatility — it is a useful starting model that highlights the difficulties in, and suggest appropriate methods for, valuing future outcomes of claims in the face of parameter uncertainty.

We assume in our model that the observed market price time series of the equity fully reflects all currently available information concerning the individual firm, the market interrelationships that exist between it and other firms, and any conditions related to the state of the economy. Further, market agents are assumed to actively process information on the equity — though they may have different evolving information sets that are at their disposal which they choose to use in-part or fully. Informally speaking, agents are well-aware of the occurrence of past and future unanticipated changes in volatility, and so they may only use a subset of their available information set to predict the current and future realized volatility.

Our intent is to use the canonical example of GBM to highlight the effects of parameter estimation error — a source of randomness that permeates all valuation models, and especially those depicting trades in thin markets, but has been given little attention in the practice of quantitative finance, e.g., Mykland [45]. In addition to providing new results on estimation-dependent contingent claim values, our working model is suggestive of approaches that can be pursued to extend the study of estimation risk to other more-complex set-ups — perhaps even to those regimes that incorporate economic behavior that is subjected to a set of intermittent volatility shocks drawn from some well-specified probability law.

In terms of method, the precursors to this paper are the articles of Boyle and Ananthanarayanan [11], Butler and Schachter [15], and Ncube and Satchell [48]. All three papers consider a vanilla European call and use the so-called "law" of the unconscious statistician-quant [5] (LUQ) to integrate the classic Black–Scholes–Merton (BSM) call formula [9, 41] with respect to a probabilistic stand-in for squared volatility. The substitute they choose, on frequentist grounds, is a chi-squared (χ^2_ν) random variable having ν degrees of freedom (df). The end product of this process results in a call value that in some, but not all, cases correctly incorporates estimation risk. In other words, the BSM formula for a call — or for that matter any related formula depending on the underlying — is treated as a random variable (of squared volatility). In turn, its expectation is calculated, where the expectation operator is governed by the particular measure reflecting the agent's current gathered information.

At the implementation stage, for the above-cited papers, the corresponding equity log-returns process is used to estimate volatility and the calculated value is plugged into an option formula. Unfortunately, this method is at least one step removed from the basic object that most economic agents view as susceptible to changes in volatility — namely, the equity process itself. The important point to note in all this discussion is that if one chooses to apply the LUQ at some step, then one should at least be aware of the fact that the stage at which it is invoked can make a difference in the subsequent phases of valuation and hedging via the option Greeks (sensitivities).

The underpinning result in [48] is their Lemma 2.1 which itself hinges on a well-known result concerning the estimator of the variance of an independent and identically distributed (i.i.d.) sequence of $\mathrm{Nor}(0, \sigma^2)$ random variables. We use this result as well, but in an entirely different spirit. The BSM option formula is typically asserted as fundamentally correct, and then for calibration purposes, a fudge factor is appended to the volatility specification so as to improve the prevailing fit-to-market. The problem is that the world in which economic agents reside is much more complicated then the BSM assumptions allow for, and encompasses many additional uncertainties. Our paper moves a little closer to addressing this problem. Its approach is from first principles, initiated at the level of the underlying, with consequently sharper and more-general results that can also be applied to other types of Lévy processes. The central tenet of the paper — an appropriate valuation and hedging strategy that deals with parameter risk — is applied to an assortment of European vanilla and exotic option types — those having a closed-form representation and those lacking an explicit formula.

In a framework purporting to depict economic behavior, it seems natural to consider the cumulative distribution function (c.d.f.) of the agent-perceived equity process, and then apply all the available pricing and hedging machinery. This is a subtle, but crucial point. In the forerunners to our paper, the sole source of randomness that agents deal with comes ex post via the estimator of volatility. In our a priori set-up there is also — in the eyes of the decision maker — the basic primary randomness associated with the future values of GBM, *that is in turn* subjected to the additional *perceived* risk emanating from the attached volatility estimator. As we prove, our formulation leads to an unbiased expected value for the underlying equity price. In fact, subject to the law governing the estimator, the conditional expected value of the equity is in accord with the expectation obtained under a martingale measure — in this case, the so-called risk-neutral measure. Compare this to applying the LUQ to each and every financial claim. First of all, there are European contingent claims that *cannot* be priced via the LUQ, but are amenable to valuation by our approach, e.g., a call on an arithmetic average of the underlying equity. Second, by using the LUQ, all sorts of *bias* may arise due to the nonlinearities of the underlying instruments and a fortiori their Greeks. On the other hand, a distinguishing feature of our model is that it does not suffer the layered nature of applying — if even possible — the LUQ on a piecemeal basis to an assortment of financial instruments written on an underlying. The approach we pursue represents one way by which an individual agent, who attempts to value and hedge a contingent claim in the real world, comes to grips with Knight's [34] dichotomy between uncertainty and risk. According to Knight, in the former case, economic agents are unable to assign probabilities to particular events, whereas in the latter situation, they can attach probabilities to these events.

Here is how the chapter is organized. In §2.2, we review "indirect" and "direct" methods for estimating the volatility associated with the underlying GBM process. §2.3 deals with the consequences of the direct estimation method. In particular, we highlight its effects on the perceived valuation and hedging functions for a variety of European options. In the process we make use of available, but often neglected, tools that aid the implementation of pricing and hedging. §2.4 gives conclusions and offers suggestions for future research. The proofs of the chapter's main results are in Appendix §2.5.

2.2 Basics

This section reviews two basic opposing methodologies to the problem of estimating volatility, and along the way establishes the notation used throughout our paper. In order to focus attention on valuation risk induced by parameter estimation in a simple yet reasonably sophisticated setting, we use the well-accepted workhorse of mathematical finance, the GBM constant-coefficients model of the price of an equity,

$$S(t;\sigma) \;\equiv\; s\,\exp\left\{\left(\mu-\tfrac{\sigma^2}{2}\right)t+\sigma\mathcal{W}(t)\right\} \;\sim\; s\exp\left\{\mathrm{Nor}\left(\left(\mu-\tfrac{\sigma^2}{2}\right)t,\sigma^2 t\right)\right\},\; t\geq 0, \qquad (1)$$

where $s \equiv S(0;\sigma)$ is the (known) initial price; $\mu \in \mathbb{R}$ and $\sigma > 0$ represent drift and volatility parameters characterizing the empirical market measure of the GBM process; and $(\mathcal{W}(t),\, t \geq 0)$ is a standard BM process. Equation (1) specifies that the stock price $S(t;\sigma)$ is lognormally distributed, i.e., its probability density function (p.d.f.) is

$$f_{S(t;\sigma)}(y) \;\equiv\; \frac{1}{y\sigma\sqrt{t}}\,\phi\left(\frac{\ell\mathrm{n}(\frac{y}{s})-(\mu-\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right), \quad y>0,$$

where $\phi(\cdot)$ is the standard normal p.d.f.

2.2.1 Indirect Estimation of σ

The indirect approach uses *implied* volatility [55] as an estimate of σ. Though no formulary explanation is attached to precisely how implied volatility comes to be what it is, implied volatility is somehow exactly "discerned" by surveying a liquid market in options written on an underlying asset. In our examples, we primarily consider European call options; analogous results apply to put options. The standard European call — often referred to as a vanilla — is a contract dependent on the current equity value s, that permits its owner, who pays an up-front fee for the privilege, to purchase the underlying asset at a pre-agreed strike price k, at a pre-determined expiry instant T time units in the future. With $\boldsymbol{v} \equiv (s,k,T)$ denoting this discernible vector of market data, the contract has value at expiry $C(\boldsymbol{v};\sigma) \;\equiv\; (S(T;\sigma)-k)^+$, where $x^+ = \max(x,0)$.

By its very nature, the contingent claim $C(\boldsymbol{v};\sigma)$ is a random variable whose current value is reflected by the observed market price of the call, say c_{m}, which is thought to incorporate the beliefs of market participants concerning the inherent variability of the underlying tradable over time interval $[0,T]$. In particular, at $t=0$, given the option's market price c_{m} and the known values s, T, r, and k, the implied volatility [55] is the number obtained by solving the implicit equation

$$c_{\mathrm{m}} = s\,\Phi(z_{+}) - k\,\mathrm{e}^{-rT}\,\Phi(z_{-}) = \mathrm{e}^{-rT}\mathrm{E}[C(\boldsymbol{v};\sigma)] = \mathrm{e}^{-rT}\int_0^\infty (y-k)^+ f_{S(T;\sigma)}(y)\,\mathrm{d}y \qquad (2)$$

for σ, where r is the fixed risk-free interest rate, $\Phi(\cdot)$ is the standard normal c.d.f., and

$$z_{\pm} \;\equiv\; \frac{\ell\mathrm{n}\left(\frac{s}{k}\right) + \left(r \pm \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}. \tag{3}$$

The term immediately to the right of the first equality in Equation (2) is the classic formula of BSM [9] giving the value of a call option. It is interpreted as the present value of $\mathrm{E}[C(\boldsymbol{v};\sigma)]$ at time 0, the expectation being taken with respect to the risk-neutral measure, i.e., by imposing the replacement of r for μ in Equation (1) — this interpretation is precisely the second equality in Equation (2).

Since all parameter values except σ are known, one can ostensibly avoid problems associated with utilizing historical data in the estimation of volatility by using a numerical solver to "discover" σ from Equation (2). However, this indirect strategy often introduces ambiguity for what σ is, as evidenced by the fact that different expiry dates and strike rates provide different values for what is supposed to be the same volatility. In fact, recognizable patterns — so-called "smiles" or "smirks" — are generated from an option chain when plotting implied σ against expiry dates [55]. Some of the standard rationalizations for this artifact are that the model formulae are incorrect representations of economic behavior or that the market lacks sufficient liquidity at all strike-expiry combinations. As a result, much effort has been expended on tweaking various volatility specifications to better fit the formulae to the market data, at the cost of introducing additional (neglected) estimation risk. We next discuss, as a complementary approach, the simplest explicit accounting of estimation risk in contingent claim valuation formulae.

2.2.2 Direct Estimation of σ

The idea behind the results in this section is to estimate σ using data available in an "estimation period" occurring before the present time, say during $[-n, 0]$; and then at time $t = 0$, use the estimate of σ to obtain the present discounted value of the option and its sensitivities (Greeks) — incorporating the induced estimation risk — attached to the future expiry date T.

With estimation of σ in mind, suppose we model the equity price during the estimation period $[-n,\, 0]$ analogously to (1), i.e.,

$$\widetilde{S}(t;\sigma) \;\equiv\; \widetilde{S}(-n;\sigma)\,\exp\Big\{\Big(\mu - \tfrac{\sigma^2}{2}\Big)(n+t) + \sigma\widetilde{\mathcal{W}}(n+t)\Big\}, \quad -n \le t \le 0,$$

where $\widetilde{S}(-n;\sigma)$ is the equity price at time $-n$ and $(\widetilde{\mathcal{W}}(n+t),\, -n \le t \le 0)$ is a standard BM. A defining property of BM implies that any increments from the estimation segment of the underlying BM are independent of the post-estimation segment $(\mathcal{W}(t),\, t \ge 0)$. With no loss in generality, divide the interval $[-n,\, 0]$ into n equal increments, from which we obtain the GBM process log-returns,

$$R_i \;\equiv\; \ell\mathrm{n}\Big(\frac{\widetilde{S}(-n+i;\sigma)}{\widetilde{S}(-n+i-1;\sigma)}\Big) \;=\; \mu - \tfrac{\sigma^2}{2} + \xi_i, \quad \text{for } i = 1, 2, \ldots, n,$$

where $\xi_i \;\equiv\; \sigma\big[\widetilde{\mathcal{W}}(i) - \widetilde{\mathcal{W}}(i-1)\big]$ for $i = 1, 2, \ldots, n$. By independent increments of BM, $R_1, R_2, \ldots, R_n$ are i.i.d. $\mathrm{Nor}(\mu - \frac{\sigma^2}{2}, \sigma^2)$ random variables. The task of estimating σ^2 is then standard under the GBM model, for in this case, we use the sample variance of the R_i's as the point estimator, i.e.,

$$\widehat{\sigma}_n^2 \;\equiv\; \frac{1}{n-1}\sum_{i=1}^{n}(R_i - \bar{R}_n)^2 \;=\; \frac{1}{n-1}\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2 \;\sim\; \frac{\sigma^2\chi^2_{n-1}}{n-1}, \tag{4}$$

where $\bar{R}_n \;\equiv\; \sum_{i=1}^{n} R_i/n$ and $\bar{\xi}_n \;\equiv\; \sum_{i=1}^{n} \xi_i/n$ are the appropriate sample means. Thus, $\mathrm{E}[\widehat{\sigma}_n^2] = \sigma^2$, so that $\widehat{\sigma}_n^2$ is unbiased for σ^2. In addition, it is easy to obtain the related result $\mathrm{E}[\widehat{\sigma}_n] = \sigma\sqrt{\frac{2}{n-1}}\,\Gamma(\frac{n}{2})/\Gamma(\frac{n-1}{2})$, where $\Gamma(\cdot)$ is the gamma function; this expression converges to σ fairly quickly as n increases. For instance, for $n = 3$, 4, 5, 10, and 100, we have $\mathrm{E}[\widehat{\sigma}_n]/\sigma$ $= 0.564$, 0.921, 0.940, 0.973, and 0.998, respectively.

For completeness, we note that an expression related to Equation (4) is the *realized* variance [55], $\widehat{V}_n^2 \;\equiv\; \sum_{i=1}^{n} \xi_i^2$, and can instead be used in data sampled at higher frequencies, i.e., data with time ticks given in seconds or minutes, where $\mu = 0$ is assumed. Realized variance is related to the quadratic variation process associated with GBM and is assumed to approximately follow the law $\sigma^2\chi^2_n/n$, with $\widehat{V}_n$ being viewed as an estimator for volatility.

2.3 Consequences of Estimating σ

This section addresses the consequences encountered in valuation and hedging when we incorporate the estimator $\widehat{\sigma}_n$ in the classic BSM valuation model.

2.3.1 Results Concerning the Underlying Asset

Our first goal is to derive the distribution of the random variable $S(t;\widehat{\sigma}_n)$ — the equity price at time t reflecting the estimation risk encompassed in $\widehat{\sigma}_n$. The following results provide expressions for the post-estimation c.d.f. and p.d.f. of the equity process.

Lemma 1 Suppose that $\widehat{\sigma}^2$ is an estimator of σ^2 that has p.d.f. $f_{\widehat{\sigma}^2}(\cdot)$ and is independent of the underlying BM process $\mathcal{W}(t)$. The c.d.f. and p.d.f. of $S(t;\widehat{\sigma})$ are

$$F_{S(t;\widehat{\sigma})}(y) \;=\; \int_0^\infty \Phi\left(\frac{\ell\text{n}(\frac{y}{s}) - \left(\mu - \frac{w}{2}\right)t}{\sqrt{wt}}\right) f_{\widehat{\sigma}^2}(w)\,\mathrm{d}w, \quad y > 0, \tag{5}$$

and

$$f_{S(t;\widehat{\sigma})}(y) \;=\; \int_0^\infty \frac{1}{y\sqrt{wt}}\,\phi\left(\frac{\ell\text{n}(\frac{y}{s}) - \left(\mu - \frac{w}{2}\right)t}{\sqrt{wt}}\right) f_{\widehat{\sigma}^2}(w)\,\mathrm{d}w, \quad y > 0, \tag{6}$$

respectively.

By construction the direct estimator $\widehat{\sigma}_n^2$ is distributed as a $\sigma^2\chi^2_{n-1}/(n-1)$ random variable that it is independent of $(\mathcal{W}(t),\, t \geq 0)$ (recall that $\widehat{\sigma}_n$ consists of data from time interval $[-n,\, 0]$). Then we immediately obtain the following computationally useful corollary, where $f_{\chi^2_{n-1}}(\cdot)$ is the χ^2_{n-1} p.d.f.

Corollary 1 The c.d.f. and p.d.f. of $S(t;\widehat{\sigma}_n)$, $n \geq 2$, are

$$F_{S(t;\widehat{\sigma}_n)}(y) \;=\; \frac{n-1}{\sigma^2}\int_0^\infty \Phi\left(\frac{\ell\text{n}(\frac{y}{s}) - \left(\mu - \frac{w}{2}\right)t}{\sqrt{wt}}\right) f_{\chi^2_{n-1}}\!\left(\tfrac{(n-1)w}{\sigma^2}\right)\mathrm{d}w, \quad y > 0, \tag{7}$$

and

$$\begin{aligned}
f_{S(t;\widehat{\sigma}_n)}(y) \;&=\; \frac{n-1}{\sigma^2}\int_0^\infty \frac{1}{y\sqrt{wt}}\,\phi\left(\frac{\ell\text{n}(\frac{y}{s}) - \left(\mu - \frac{w}{2}\right)t}{\sqrt{wt}}\right) f_{\chi^2_{n-1}}\!\left(\tfrac{(n-1)w}{\sigma^2}\right)\mathrm{d}w, \quad y > 0 \\
&=\; \frac{K\mathrm{e}^{-a_0(y)/2}}{y}\int_0^\infty \exp\left\{-\left(\frac{a_1(y)}{w} + a_2 w\right)\right\} w^{\frac{n-4}{2}}\,\mathrm{d}w, \quad y > 0 \qquad (8) \\
&=\; \frac{\mathrm{e}^{\frac{r\tau}{2}}}{\Gamma\left[\frac{n-1}{2}\right]}\left(4(n-1)\tau + \sigma^2\tau^2\right)^{\frac{2-n}{4}}\sqrt{\tfrac{(n-1)^{n-1}}{\pi\sigma^n\tau}}\,\tfrac{1}{y}\sqrt{\tfrac{s}{y}}\left|r\tau - \ell\text{n}\left(\tfrac{y}{s}\right)\right|^{\frac{n-2}{2}} \\
&\qquad \times\, \mathcal{K}_{\frac{2-n}{2}}\left[\sqrt{\tfrac{n-1}{\sigma^2\tau} + \tfrac{1}{4}}\left|r\tau - \ell\text{n}\left(\tfrac{y}{s}\right)\right|\right], \quad y \geq 0, \qquad (9)
\end{aligned}$$

where we define in Equation (8) the functions $a_0(y) \equiv \ell\text{n}(\frac{y}{s}) - \mu t$ and $a_1(y) \equiv \frac{a_0^2(y)}{2t}$, and the positive numbers $a_2 \equiv \frac{t}{8} + \frac{n-1}{2\sigma^2}$ and $K \equiv \left(\frac{n-1}{2\sigma^2}\right)^{\frac{n-1}{2}}\left(\Gamma(\frac{n-1}{2})\sqrt{2\pi t}\right)^{-1}$; additionally, in Equation (9) the term $\mathcal{K}_\alpha[\beta]$ denotes a modified Bessel function [25] of the second kind with parameters α and β.

The above results are the laws viewed by economic agents as systematically quantifying their imperfect knowledge concerning a key equity market characteristic — namely, volatility. In fact, Lemma 1 is just an application of the law of total probability. At the next and succeeding stages, these precise descriptions, i.e., the c.d.f. or p.d.f., are further incorporated into agent decision rules concerning valuation and the choice of hedging strategies.

Example 1 Figure 3(a) depicts the post-estimation p.d.f.'s $f_{S(t;\widehat{\sigma}_n)}(\cdot)$ for the case $t = 1/2$, $s = 10$, $\mu = 0.05$, and $\sigma = 1$ using estimates $\widehat{\sigma}_n$ based on $n = 3$, 4, 10, 30, and BSM. For further increases in n, the distinction between post-estimation and BSM becomes inconsequential. On the other hand, we see that for small n, the p.d.f.'s differ substantially from the limiting lognormal density. Figure 3(b) plots the p.d.f.'s for the case $n = 4$, $t = 1$, $s = 10$ and $\mu = 0.05$, for true values of $\sigma = 1/2$, $3/4$, and 1. Evidently, the value of σ significantly impacts the shape of the density; for example, the density with $\sigma = 1$ has relatively heavier tail behavior.

The next corollary, subject to mild restrictions, gives the moments for the density $f_{S(t;\widehat{\sigma})}(\cdot)$.

Corollary 2 Under the conditions of Lemma 4, the jth moment of $S(t;\widehat{\sigma})$ is

$$\mathrm{E}[S^j(t;\widehat{\sigma})] \quad = \quad (s\mathrm{e}^{\mu t})^j M_{\widehat{\sigma}^2}\left(\frac{t(j^2-j)}{2}\right),$$

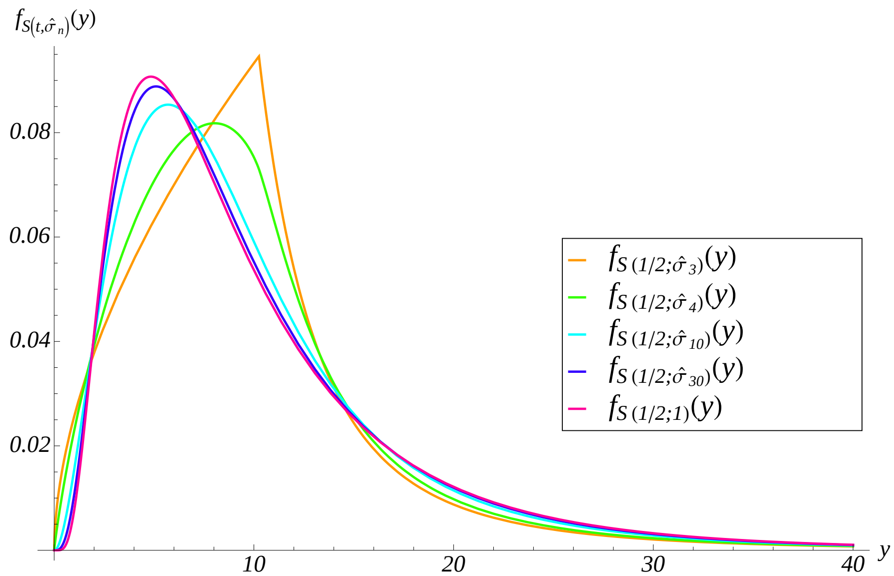
where $M_{\widehat{\sigma}^2}(\cdot)$ is the moment generating function (m.g.f.) of $\widehat{\sigma}^2$.

Notice that for *any* estimator $\widehat{\sigma}$ satisfying the conditions of Lemma 4, the estimation-augmented underlying inherits the expected value property of the lognormally distributed asset price at time t, i.e.,

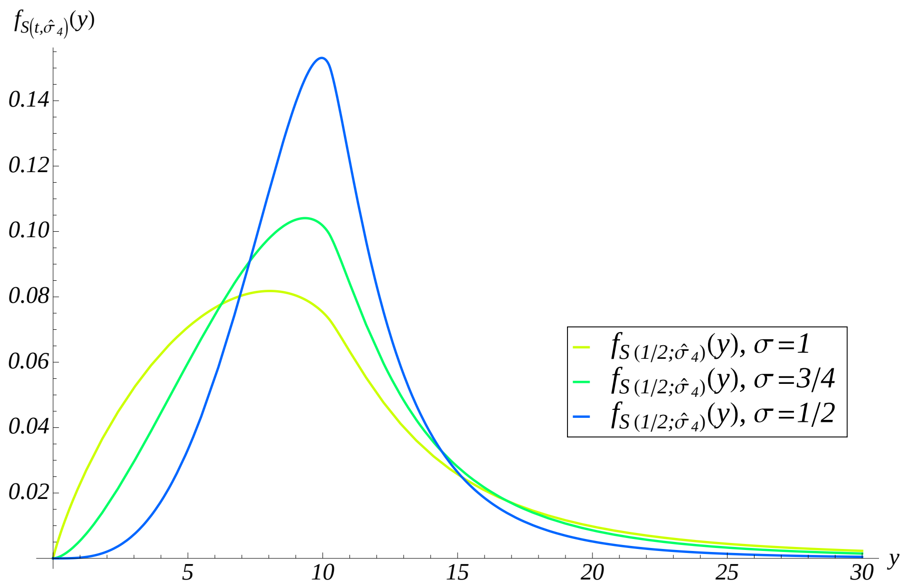
$$\mathrm{E}[S(t;\widehat{\sigma})] = s\mathrm{e}^{\mu t}.$$

In particular, with $\mu = r$, this is the no-arbitrage *forward* price of the underlying and is independent of the volatility estimation period.

Since the m.g.f. of the χ^2_ν distribution is $M_{\chi^2_\nu}(y) = (1-2y)^{-\nu/2}$ for $y < 1/2$, we easily obtain moment results for the direct variance estimator $\widehat{\sigma}^2_n$.



(a) $s = 10$; $t = 1/2$; $n = 3$, 4, 10, 30, and BSM; $\sigma = 1$; and $\mu = 0.05$



(b) $s = 10$; $t = 1/2$; $n = 4$; $\sigma = 1/2$, $3/4$, 1; and $\mu = 0.05$

Figure 3: A cornucopia of $f_{S(t;\widehat{\sigma}_n)}(\cdot)$ p.d.f.'s.

Corollary 3 If $j \geq 1$ and $n \geq \max\{2, 1 + \sigma^2 t(j^2 - j)\}$, then the moments of $S(t; \widehat{\sigma}_n)$ are

$$\mathrm{E}[S^j(t; \widehat{\sigma}_n)] \;=\; (s\mathrm{e}^{\mu t})^j \left(1 - \frac{\sigma^2 t(j^2 - j)}{n-1}\right)^{-\frac{n-1}{2}}.$$

From Corollary 3, the variance of the estimation augmented equity price is

$$\mathrm{Var}[S(t; \widehat{\sigma}_n)] \;=\; (s\mathrm{e}^{\mu t})^2 \Big[\big(1 - \tfrac{2\sigma^2 t}{n-1}\big)^{-\frac{n-1}{2}} - 1\Big], \tag{10}$$

a value dependent on n.

Next we present an exact recipe for simulating from the post-estimation GBM process. The method is needed in order to implement a subsequent valuation example. The following pseudo-code provides one simulated realization of the path of the underlying $(S(t; \widehat{\sigma}_n),\, t \geq 0)$ at times $t = 0, T/m, 2T/m, \ldots, T$, where $m \geq 1$ can be regarded as a "mesh" factor.

Algorithm 1 Simulating a Sample Path of the Post-Estimation Underlying

1. Initialize $n \geq 2$; σ; r or μ; T; s; m; and $\mathcal{W}(0) = 0$.

2. Generate $\widehat{\sigma}_n^2 \leftarrow \frac{\sigma^2}{n-1}\, \chi^2_{n-1}$.

3. Generate a standard Brownian motion sample path: For $i = 1, 2, \ldots, m$, set

$$\mathcal{W}(\tfrac{iT}{m}) \leftarrow \mathcal{W}(\tfrac{(i-1)T}{m}) + \sqrt{\tfrac{T}{m}}\, Z_i,$$

   where $Z_1, Z_2, \ldots, Z_m$ are i.i.d. $\mathrm{Nor}(0,1)$ (and independent of $\widehat{\sigma}_n^2$).

4. For $i = 1, 2, \ldots, m$, set $S(\frac{iT}{m}; \widehat{\sigma}_n) \leftarrow s \exp\!\big((\mu - \frac{1}{2}\widehat{\sigma}_n^2)\,\frac{iT}{m} + \widehat{\sigma}_n\, \mathcal{W}(\frac{iT}{m})\big)$.

When choosing to generate a sample path of $(S(t; \sigma),\, t \geq 0)$, we skip Step 2 and then use σ instead of $\widehat{\sigma}_n$ throughout.

Example 2 The pth quantile of a random variable X with c.d.f. $F(x)$ is defined as $F^{-1}(p) \equiv \inf\{x : p \leq F(x)\}$. Figure 4 is a sequence of quantile-quantile (Q–Q) plots — each with a superimposed 45° line — allowing for a comparison of how well the post-estimation c.d.f.'s we developed in Corollary 4 conform to the lognormal c.d.f. of GBM. Two c.d.f.'s describe the same distribution if the Q–Q plot coincides with the diagonal line. Figure 4(a) is used as a reference for the post-estimation cases. For (a), we generated two independent sets of 10^5

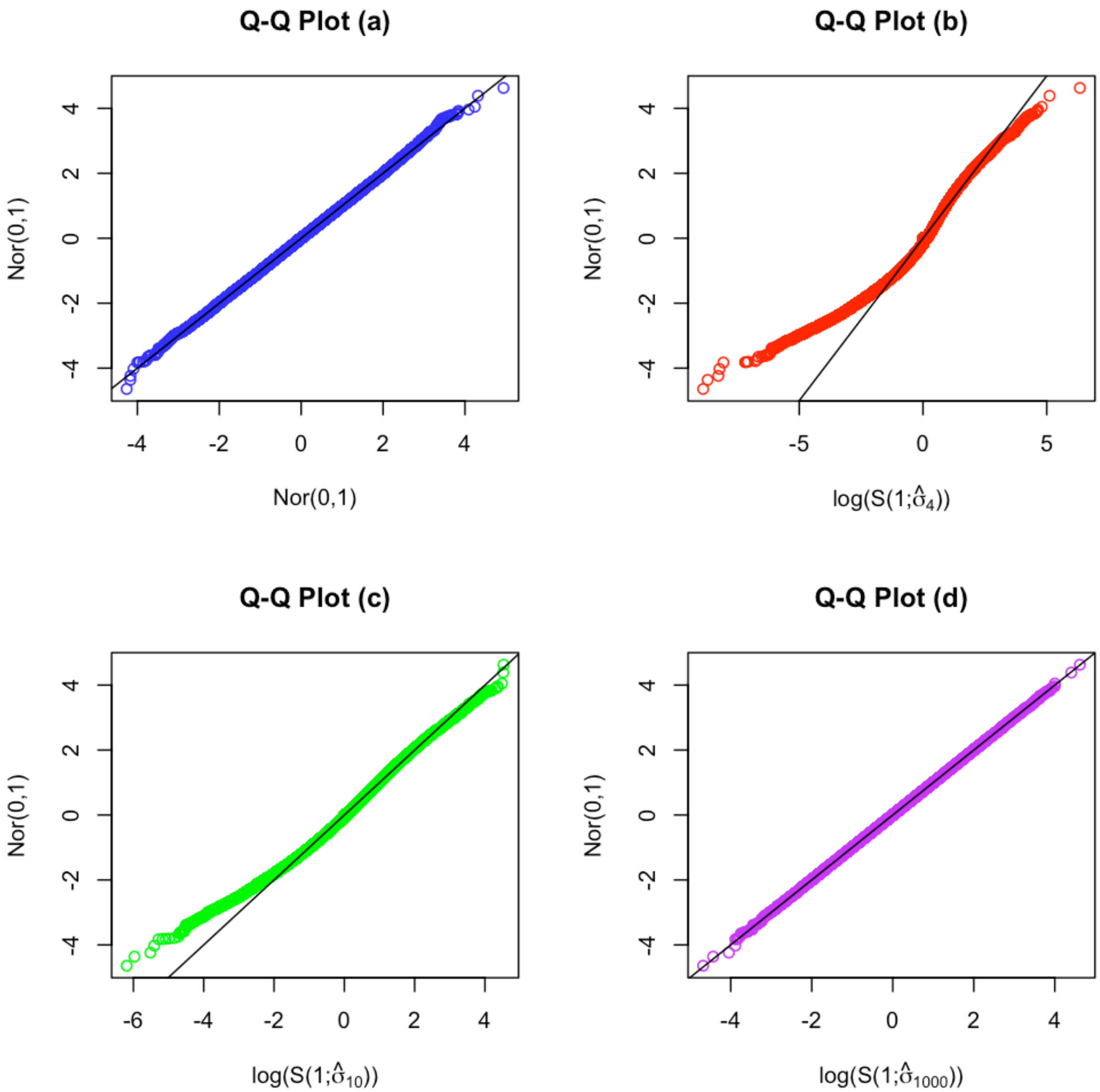


Figure 4: Q–Q Plots of $\ell\text{n}(S(1; \widehat{\sigma}_n))$

replications of GBM with $(\mu, \sigma) = (1/2, 1)$ and $t = 1$. It follows that the log of each sample draw is $\mathcal{W}(1) \sim \text{Nor}(0,1)$. Not surprisingly, excellent conformity exists between these two independent samples. In part (b), we use the initial $\text{Nor}(0,1)$ series of the reference figure, and compare it to the logs of a sample of 10^5 replications from the post-estimation c.d.f. with $n = 4$. Clearly, the post-estimation c.d.f. deviates from lognormal. Maintaining 10^5 replications, part (c) has a post-estimation c.d.f. with $n = 10$, exhibiting less-severe deviation from a lognormal c.d.f. Similarly, part (d) compares the post-estimation c.d.f. with $n = 1000$ to the lognormal case. We can see that this post-estimation c.d.f. is, for all practical purposes, equal to the limiting lognormal.

2.3.2 Results Concerning the Underlying Log-Returns

The next lemma gives the post-estimation c.d.f. and p.d.f. of the continuous version log-returns process, and is analogous to Corollary 4. First of all, define the post-estimation unconditional log-return over any time interval $[t, t + \Delta]$ as

$$R(t, \Delta; \widehat{\sigma}_n) \equiv \left(\mu - \tfrac{\widehat{\sigma}_n^2}{2}\right)\Delta - \widehat{\sigma}_n(\mathcal{W}(t + \Delta) - \mathcal{W}(t)), \quad t \geq 0,\, n \geq 2. \tag{11}$$

Lemma 2 The c.d.f. and p.d.f. of $R(t, \Delta; \widehat{\sigma}_n)$, $n \geq 2$, are

$$F_{R(t,\Delta;\widehat{\sigma}_n)}(x) = \frac{n-1}{\sigma^2} \int_0^\infty \Phi\left(\frac{x - \left(\mu - \frac{w}{2}\right)\Delta}{\sqrt{w\Delta}}\right) f_{\chi^2_{n-1}}\left(\tfrac{(n-1)w}{\sigma^2}\right) \mathrm{d}w, \; x \in \mathbb{R}, \tag{12}$$

$$\begin{aligned} f_{R(t,\Delta;\widehat{\sigma}_n)}(x) &= \frac{n-1}{\sigma^2} \int_0^\infty \phi\left(\frac{x - \left(\mu - \frac{w}{2}\right)\Delta}{\sqrt{w\Delta}}\right) f_{\chi^2_{n-1}}\left(\tfrac{(n-1)w}{\sigma^2}\right) \mathrm{d}w, \; x \in \mathbb{R} \\ &= \frac{\mathrm{e}^{-\frac{b_0(x)}{2}} (n-1)^{\frac{n-1}{2}} \left(\frac{b_1\Delta}{b_0^2(x)\sigma^2}\right)^{\frac{2-n}{4}} \mathcal{K}_{1-\frac{n}{2}}\left[\frac{\sqrt{b_0^2(x) b_1}}{2\sigma\sqrt{\Delta}}\right]}{\sqrt{\pi\Delta}\,\sigma^{n-1}\Gamma\left(\frac{n-1}{2}\right)}, \end{aligned} \tag{13}$$

where $b_0(x) \equiv x - \mu\Delta$ and $b_1 \equiv 4(n-1) + \Delta\sigma^2$.

A raison d'être of the current chapter is the need to directly simulate the pre-estimation $(S(t; \sigma),\, t \geq 0)$ and post-estimation $(S(t; \widehat{\sigma}_n),\, t \geq 0)$ processes in order to value and hedge certain contingent securities. The addition of the log-returns process to our repertoire allows us to extend the post-estimation results to Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) metrics [51]. We do not further pursue these calculations here. However, we do provide two exact recipes for sampling from the distribution characterized by Lemma

2. The algorithms are useful when calculating and stress testing the VaR and CVaR metrics associated with particular financial portfolios. The two methods for simulating i.i.d. log-returns, denoted by $R(t_i, \Delta; \widehat{\sigma}_n)$, $i = 1, 2, \ldots, m$, based on m time intervals $[t, t+\Delta]$ of length $\Delta = T/m$, are given in the following algorithm.

Algorithm 2 Simulating a Sample Path of the Post-Estimation Log-Returns Process

A. This method follows from the definition in (11). Generate

$$R(\tfrac{iT}{m}, \Delta; \widehat{\sigma}_n) \leftarrow \big(\mu - \tfrac{\widehat{\sigma}_n^2}{2}\big)\Delta + \widehat{\sigma}_n \sqrt{\Delta}\, Z_i, \quad i = 1, 2, \ldots, m.$$

B. The second method is suited for the case where one desires joint sample paths of the underlying equity path and its log-returns process, and it is useful if one has already invoked Algorithm 1.

$$R(\tfrac{iT}{m}, \Delta; \widehat{\sigma}_n) \leftarrow \ell\mathrm{n}\left(\frac{S\big(i\Delta; \widehat{\sigma}_n\big)}{S\big((i-1)\Delta; \widehat{\sigma}_n\big)}\right), \quad i = 1, 2, \ldots, m.$$

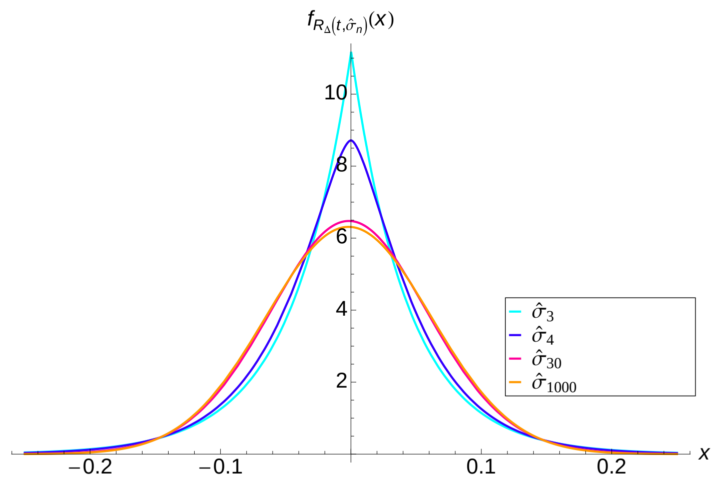
With the aid of the characteristic function $\Psi_{R(t,\Delta;\widehat{\sigma}_n)}(\cdot)$ for the post-estimation log-returns process, or for that matter directly, the moments of $R(t, \Delta; \widehat{\sigma}_n)$ can be calculated in explicit form. The columns of Table 1 list and compare the summary statistics — exact and simulated via Algorithm 2. We assume that the data gathered is of the high-frequency type, and hence it is reasonable to set $\mu = 0$. In order to obtain reasonable precision for third and fourth moments, our simulation uses the brute force method consisting of a sample size of 10^7. In particular, the first four moments $m_j \equiv \mathrm{E}[R^j(t, \Delta; \widehat{\sigma}_n)]$, $j = 1, 2, 3, 4$, are $m_1 = \frac{\Delta\sigma^2}{2}$, $m_2 = 2m_1 + \frac{n+1}{n-1}m_1^2$, $m_3 = 6\frac{n+1}{n-1}m_1^2 + \frac{n+3}{n-1}m_1^3$, and $m_4 = 12\frac{n+1}{n-1}m_1^2 + 12\frac{(n+1)(n+3)}{(n-1)^2}m_1^3 + \frac{(n+1)(n+3)(n+5)}{(n-1)^3}m_1^4$. Whence, the summary measures (dependent on n) are [30] $\mathbb{V} = m_2 - m_1^2$ (variance), $\mathbb{S} = (m_3 - 3m_2m_1 + 2m_1^3)/\mathbb{V}^{3/2}$ (skewness), and $\mathbb{K} = (m_4 - 4m_3m_1 + 6m_2m_1^2 - 3m_1^4)/\mathbb{V}^2$ (kurtosis). Note the bias in the first moment of the post-estimation log-returns. All other shape measures for the post-estimation case have as limits those of the normal c.d.f.

In Figure 5(a) which accompanies Table 1, we illustrate several post-estimation log-return densities $f_{R(t,\Delta;\widehat{\sigma}_n)}(\cdot)$ — generated numerically via Lemma 2 and instantiated by

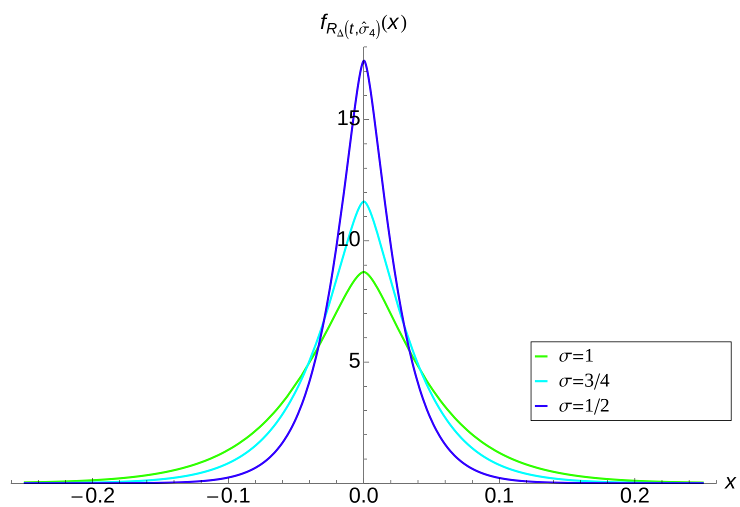
Table 1: Log-return Summary Measures: Exact and Simulated

	$F_{R(t,\Delta;\sigma)}(\cdot)$	$F_{R(t,\Delta;\widehat{\sigma}_n)}(\cdot)$	$n=3$	$n=4$	$n=10$	$n=30$	$n=1000$
m_1	$-\frac{\sigma^2\Delta}{2}$	$\frac{\sigma^2\Delta}{2}$	-0.0002	-0.0002	-0.0002	-0.0002	-0.0002
$\mathbb{V}$	$-2m_1$	$2m_1(1+\frac{m_1}{n-1})$	0.0005	0.0005	0.0005	0.0005	0.0005
$\mathbb{S}$	0	$4\sqrt{\frac{m_1}{n-1}}\frac{3(n-1)+2m_1}{[2(n-1)+2m_1]^{3/2}}$	-0.0047	-0.0011	0.0119	0.0158	0.0180
$\mathbb{K}$	3	$\frac{3(n-1)^2(n+1)+(n+3)[6(n-1)+(2n+1)m_1]m_1}{(n-1)(m_1+n-1)^2}$	4.9931	4.4940	3.5979	3.1989	3.0088

$\sigma=1$; $\Delta=1/1750$; $\mu=0$



(a) $n=3$, 4, 10, 30, and BSM; $\sigma=1$; $\Delta=1/1750$; $\mu=0$.



(b) $n=4$; $\sigma=1/2$, 3/4, 1; $\Delta=1/1750$; $\mu=0$.

Figure 5: Cornucopia of $f_{R(t,\Delta;\widehat{\sigma}_n)}(\cdot)$ densities.

parameter values $\mu = 0$ and $\sigma = 1$, using estimators $\widehat{\sigma}_n$ based on $n = 3, 4, 30, 1000$. We assume for simplicity that returns are all based on 250 trading days, each covering seven hours of market activity; and so we take $\Delta = 1/1750$ and $t_i = i\Delta$. The heavy-tail behavior of the density $f_{R(t_i,\Delta;\widehat{\sigma}_n)}(\cdot)$ manifests more strongly for small n. This fact is confirmed by noting the sample kurtosis values in the last row of Table 1 and the matching p.d.f.'s in Figure 5(a). In all cases log-returns are generated via Algorithm 2. Figure 5(b) presents log-return densities for $\mu = 0$, $\Delta = 1/1750$, $n = 4$, and $\sigma = 1/2$, $3/4$, and 1. We conclude that incorporating the estimator of volatility into our analysis of the underlying equity can make a substantial difference in the tail probabilities of the respective log-return p.d.f.'s.

2.3.3 Results Concerning European Claims

This section gives a number of examples illustrating the consequences of including estimation risk within an option valuation. Our attention is directed at the contingent claims that correspond to the post-estimation c.d.f. $F_{S(T;\widehat{\sigma}_n)}(\cdot)$. All the pre- and post-estimation valuation formulae are derived under the assumption that $\mu = r$, which accords with the BSM risk-neutral measure. Rather then explore the totality of European claim structures, we confine our examples to call options — standard and exotic. These illustrate the wedge in valuations induced by known versus estimated σ. The difference in pricing, it turns out, is often significant — on the order of few basis points to several hundred basis points — though not overwhelmingly so large as to cast doubt on the underlying BSM model. Our extension of the BSM model should be viewed as a calibration more in line with reality — one that will be of concern to institutions dealing with the valuation and hedging of a portfolio marked-to-market at many billions of dollars.

2.3.3.1 Vanilla Calls and Puts

The c.d.f. of the vanilla European call option, $C(\boldsymbol{v};\sigma)$ — inclusive of the volatility estimator $\widehat{\sigma}_n$ — is given by

$$F_{C(\boldsymbol{v};\widehat{\sigma}_n)}(y) \equiv \Pr\left(C(\boldsymbol{v};\widehat{\sigma}_n) \leq y\right) = \begin{cases} 0 & \text{if } y < 0 \\ F_{S(T;\widehat{\sigma}_n)}(y+k) & \text{if } y \geq 0. \end{cases} \tag{14}$$

The call has a point probability at $y = 0$ equal to $F_{S(T;\widehat{\sigma}_n)}(k)$ — the probability of being out-of-the-money (OTM) at the time of expiry. The analogous put, $P(\boldsymbol{v};\sigma) \equiv (k - S(T;\sigma))^+$, has post-estimation c.d.f.

$$F_{P(\boldsymbol{v};\widehat{\sigma}_n)}(y) \;\equiv\; \Pr\big(P(\boldsymbol{v};\widehat{\sigma}_n) \le y\big) \;=\; \begin{cases} 0 & \text{if } y < 0 \\ 1 - F_{S(T;\widehat{\sigma}_n)}(k - y) & \text{if } y \ge 0, \end{cases}$$

likewise with a point probability for an OTM event at $y = 0$.

Now consider the present value of the call $C(\boldsymbol{v};\widehat{\sigma}_n)$. By Equation (14),

$$\begin{aligned} c(\boldsymbol{v};\widehat{\sigma}_n) &\equiv \mathrm{e}^{-rT}\mathrm{E}[C(\boldsymbol{v};\widehat{\sigma}_n)] \\ &= \mathrm{e}^{-rT}\int_k^\infty \big(1 - F_{S(T;\widehat{\sigma}_n)}(y)\big)\,\mathrm{d}y \;\equiv\; \mathrm{e}^{-rT}\int_k^\infty \bar{F}_{S(T;\widehat{\sigma}_n)}(y)\,\mathrm{d}y, \end{aligned} \tag{15}$$

which, at the very least, can be solved numerically. Similarly, for the put we have

$$p(\boldsymbol{v};\widehat{\sigma}_n) \;\equiv\; \mathrm{e}^{-rT}\mathrm{E}[P(\boldsymbol{v};\widehat{\sigma}_n)] \;=\; \mathrm{e}^{-rT}\int_0^k F_{S(T;\widehat{\sigma}_n)}(y)\,\mathrm{d}y,$$

or alternatively from the standard put–call parity relation [55],

$$p(\boldsymbol{v};\widehat{\sigma}_n) \;=\; c(\boldsymbol{v};\widehat{\sigma}_n) - s + k\mathrm{e}^{-rT}.$$

Example 3 Figure 6 plots, as a function of the current equity price s, call values $c(\boldsymbol{v};\widehat{\sigma}_n)$, $n = 3, 4, 10, 30$, and the limiting BSM formula ($n = \infty$), using $k = 10$, $T = 1/2$, $r = 0.05$, and $\sigma = 1$. Thus, for this example, the inclusion of estimation risk underprices the European call value relative to BSM, with the underpricing progressively decreasing as we move further in-the-money (ITM) or OTM. For instance, the ATM valuations are $c(\boldsymbol{v};\widehat{\sigma}_3) = 2.523$, $c(\boldsymbol{v};\widehat{\sigma}_4) = 2.624$, $c(\boldsymbol{v};\widehat{\sigma}_{10}) = 2.774$, $c(\boldsymbol{v};\widehat{\sigma}_{30}) = 2.829$, and the BSM value $c(\boldsymbol{v};\sigma) = 2.854$. Note that if prices from the post-estimation pricing schedule are input into classic BSM for the purpose of obtaining implied volatility, then one will conclude that a non-constant volatility is indicated — a *fake* smile effect — even though σ is in fact constant.

Perhaps surprisingly, in the case of a vanilla call valuation, the post-estimation case gives the same value as in the case where the LUQ is applied directly to the BSM formula given in Equation (2). In particular, the machinations in [11, 15, 48] amount to the LUQ call priced

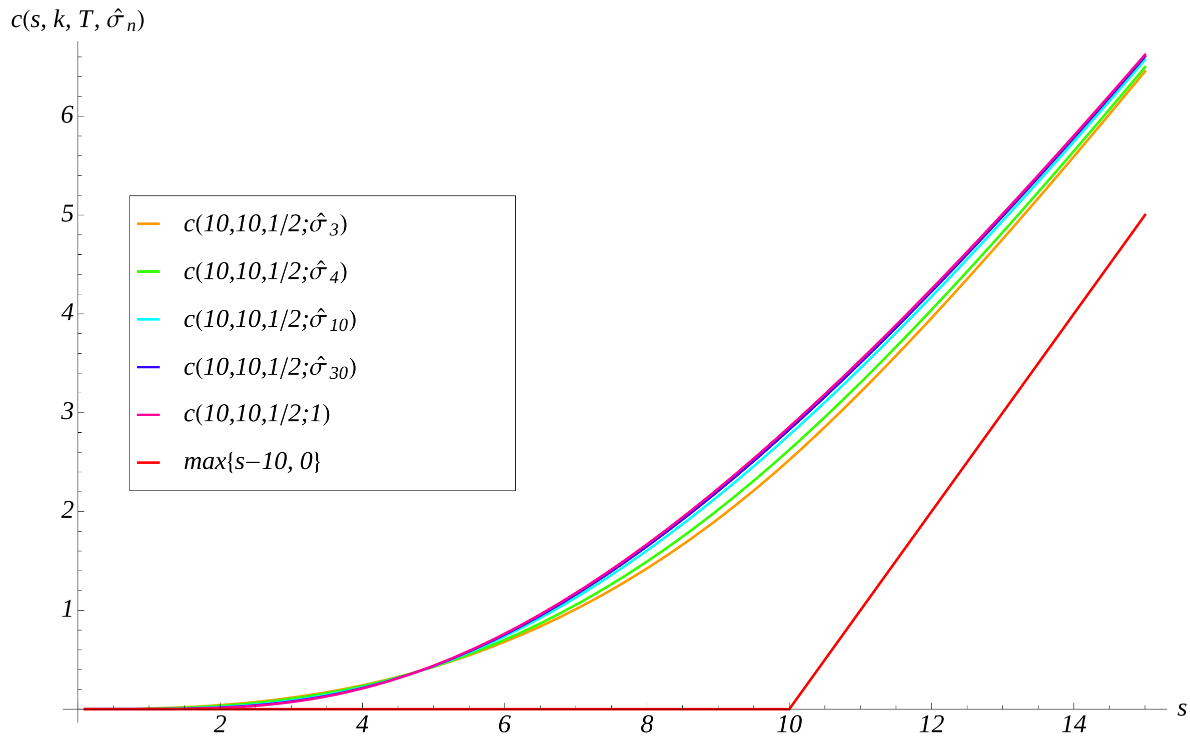


Figure 6: BSM vs. post-estimation European call valuations $c(\boldsymbol{v};\widehat{\sigma}_n)$: $\boldsymbol{v} = (s,k,T) = (10, 10, 1/2)$; $r = 0.05$; $\sigma = 1$; $n = 3, 4, 10, 30$, BSM

via $c^*(\boldsymbol{v};\widehat{\sigma}_n) \equiv \frac{n-1}{\sigma^2}\mathrm{E}\big[s\,\Phi\big(z_+(\Xi)\big) - k\,\mathrm{e}^{-rT}\,\Phi\big(z_-(\Xi)\big)\big]$, where $z_\pm(\Xi) \equiv \frac{\ell\mathrm{n}(\frac{s}{k})+(r\pm\Xi/2)T}{\sqrt{\Xi T}}$ and $\Xi \sim \frac{\sigma^2}{n-1}\chi^2_{n-1}$. We stress that the methods are not equivalent, as will be demonstrated when we consider this call's sensitivities in §2.3.3.6.

2.3.3.2 Digital Claims

The simplest of European options, and one that directly makes use of the post-estimation c.d.f., is the digital claim. The digital is predicated on the occurrence of an event $\mathcal{E}$, e.g., $\mathcal{E} = \{S(T;\sigma) > k\}$ (a digital call) or the complementary event $\bar{\mathcal{E}}$ (a digital put), and pays a "coupon" of \$1 if the event occurs. Symbolically, a digital has value

$$d(\boldsymbol{v};\widehat{\sigma}_n) \equiv \mathrm{e}^{-rT}\mathrm{E}[I_{\mathcal{E}}] = \mathrm{e}^{-rT}\,\mathrm{Pr}(\mathcal{E}),$$

where $I_{\mathcal{E}}$ is the indicator function for the event $\mathcal{E}$.

Example 4 Using Equation (14), we can calculate the OTM probability $F_{S(T;\widehat{\sigma}_n)}(k)$ of a European digital call when using $\widehat{\sigma}_n$ in place of σ. Table 2 illustrates an example for $s = 10$,

$\sigma = 1.5$, $r = 0.05$, and $T = 1/4$; and we display the resulting probabilities for strike values $k = 5, 10, 15$, and $n = 4$ and ∞, the last of which corresponds to perfect knowledge of σ. Note that for a well-ITM option ($k = 5$), the probability of being OTM at expiry is much smaller under c.d.f. $F_{S(T;\widehat{\sigma}_4)}(\cdot)$ than for the standard BSM c.d.f. $F_{S(T;\sigma)}(\cdot)$. Related OTM put probabilities can be calculated using $\bar{F}_{S(T;\widehat{\sigma}_n)}(k)$. To obtain digital option values $d(\boldsymbol{v}; \widehat{\sigma}_n)$, simply multiply the probabilities in Table 2 by $\mathrm{e}^{-rT} = 0.9876$.

Table 2: OTM Probabilities for $s = 10$, $T = 1/4$, $\sigma = 1.5$, and $r = 0.05$

k	5	10	15
$F_{S(1/4;\widehat{\sigma}_4)}(k)$	0.240	0.672	0.845
$F_{S(1/4;\sigma)}(k)$	0.442	0.694	0.813

The option sensitivities — BSM, LUQ, and our post-estimation — of the European digital can be obtained by a procedure analogous to that described in §2.3.3.6.

2.3.3.3 Barrier Options

Here we calculate the value of a digital barrier option. Define the process $M(T;\sigma) \equiv \max\{S(t;\sigma),\, 0 \le t \le T\}$, which records the maximum value of the GBM price path observed up to time T. Setting $k = B$, our choice of claim is the digital "knock-in," having payoff

$$D(\boldsymbol{v};\sigma) \equiv I_{\{M(T;\sigma)\ge B\}}. \tag{16}$$

If $S(t;\sigma)$ hits the barrier B by time T, the payoff is \$1; otherwise, the claim pays nothing.

In order to determine the fair value of $D(\boldsymbol{v};\sigma)$, we calculate the probability of the event $\{M(T;\sigma) \ge B\}$ and then discount by the risk-free rate. The details are contained in the in the appendix (Chapter 2) and depend on a bivariate Markov process constructed from a Brownian motion with drift. The c.d.f. of $M(T;\sigma)$ is (from [55])

$$\begin{aligned} F_{M(T;\sigma)}(B) &= \Pr(\max_{t\le T} S(t;\sigma) \le B) \\ &= \Phi\Big(\frac{\ell\mathrm{n}(\frac{B}{s}) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\Big) - \Big(\frac{B}{s}\Big)^{\frac{2r}{\sigma^2}-1} \Phi\Big(\frac{\ell\mathrm{n}(\frac{s}{B}) - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\Big). \end{aligned} \tag{17}$$

Thus, when the volatility is known, the fair value of the digital in Equation (16) is $d(\boldsymbol{v};\sigma) \equiv \mathrm{e}^{-rT}\bar{F}_{M(T;\sigma)}(B)$.

When σ is unknown, we employ the same logic as in Corollary 4 to obtain the digital barrier value,

$$\begin{aligned} d(\boldsymbol{v};\widehat{\sigma}_n) &= \mathrm{e}^{-rT}\,\Pr(M(T;\widehat{\sigma}_n) \geq B) \\ &= \frac{\mathrm{e}^{-rT}(n-1)}{\sigma^2}\int_0^\infty \bar{F}_{M(T;\sqrt{w})}(B) f_{\chi^2_{n-1}}\big(\tfrac{(n-1)w}{\sigma^2}\big)\,\mathrm{d}w. \end{aligned}$$

For further discussion concerning this post-estimation case, see the appendix in this chapter.

Example 5 Table 3 gives representative barrier probabilities from the two complementary c.d.f.'s $\bar{F}_{M(T;\widehat{\sigma}_4)}(B)$ $(n = 4)$ and $\bar{F}_{M(T;\sigma)}(B)$ $(n = \infty)$ for the case $T = 1$, $s = 10$, $r = 0.05$, with the true value of $\sigma = 1.5$. We see that as the barrier B is raised, the difference in values fluctuates between 80 and 140 basis points.

Table 3: Barrier Probabilities for $T = 1$, $s = 10$, $r = 0.05$, and $\sigma = 1.5$

B	11	12	13	14	15
$\bar{F}_{M(T;\widehat{\sigma}_4)}(B)$	0.846	0.720	0.617	0.532	0.462
$\bar{F}_{M(T;\sigma)}(B)$	0.854	0.732	0.631	0.546	0.475

Remark One can use simulation (e.g., Algorithm 1) as an alternative to numerically calculating barrier probabilities and accompanying fair values. However, a problem with simulation is that the hitting probabilities will be biased downward due to the discrete nature of sampling the barrier crossing. This problem can be mitigated by refining the size of the sampling mesh, but at the expense of increased computer processing time.

2.3.3.4 Other Exotics With Closed Forms

There are many non-standard options to which our methodology can, with varying degrees of difficulty, be applied. One that readily fits into our paradigm is the forward start call option [18], a component of the general class of claims called ratchets. With $0 < \bar{T} \leq T$, the forward start is

$$U(s, x, T, \bar{T}; \sigma) \equiv (S(T;\sigma) - xS(\bar{T};\sigma))^+,\ x > 0. \tag{18}$$

This can be interpreted as having a strike value $k = xS(\bar{T};\sigma)$ — now a random variable dependent on a future outcome of the underlying. In the appendix associated with this chapter, the following BSM-type of result is shown to hold:

$$u(\boldsymbol{\upsilon};\sigma) \equiv \mathrm{e}^{-rT}\mathrm{E}[U(\boldsymbol{\upsilon};\sigma)] = \mathrm{e}^{-rT}\mathrm{E}[C(T-\bar{T},ks;\sigma)], \tag{19}$$

i.e., by setting $\boldsymbol{\upsilon} = (s, x, T-\bar{T})$, and by making the replacements $T \to (T-\bar{T})$ and $k \to xs$ in Equations (2) and (3).

What happens when we substitute the estimator $\widehat{\sigma}_n$ for σ? Aside from the indicated adjustment of the parameters, the BSM formula is the same as for a vanilla option; so the forward start will have a valuation schedule similar to that of a vanilla European call — see the standard BSM formula Equation (2), or Equation (15) for the case of $\widehat{\sigma}_n$. In any case, our valuation c.d.f. has not changed — only an adjustment of parameters is required.

Another option, but now path-dependent, is the digital lookback put [55],

$$D(\boldsymbol{\upsilon};\sigma) \equiv I_{\{M(T;\sigma)-S(T;\sigma)\geq L\}},$$

where now we use the replacement $k \to L$. This digital pays \$1 if the maximum of the stock price on $[0,T]$ exceeds the terminal price by at least L. By the discussion in §2.3.3.3, the process $(M(t;\sigma)-S(t;\sigma),\ t \geq 0)$ is Markov, i.e., the transition probability depends only on the current state $(M(t;\sigma), S(t;\sigma))$. Thus, we can use our tools to price the digital lookback — post-estimation or otherwise. We note, as well, that the sensitivity results of §3.4.2 can be applied to all the contingent claims we have discussed.

With a little ingenuity, many other contingent claim types can be valued. The general idea is simple: (i) identify the Markov process determining the payoff, (ii) obtain the joint law governing the process, and (iii) use the pre- or post-estimation c.d.f. to determine the fair price.

2.3.3.5 Asian Options

In this section, we outline relevant results for a variety of Asian options, i.e., options based on certain averages of the equity price as it evolves over time. An interesting feature of some of these claim types is that no closed-form formulae exist for pricing or hedging. For these we use simulation to provide valuations.

2.3.3.5.1 Geometric Average with Known σ

We first examine the geometric average of the equity price over the time interval $[0,T]$, i.e., $S_c^G(T;\sigma) \equiv \exp\bigl(\frac{1}{T}\int_0^T \ell\mathrm{n}(S(t;\sigma))\,\mathrm{d}t\bigr)$ (continuously monitored) or $S_d^G(T;\sigma) \equiv \bigl(\prod_{i=1}^m S(t_i;\sigma)\bigr)^{1/m}$ with $0 \le t_1 < t_2 < \cdots < t_m = T$ (discretely monitored) [32]. In either case ($e = d$ or c), it is known [32] that the Asian option for $C_e^G(\boldsymbol{\upsilon};\sigma) \equiv (S_e^G(T;\sigma) - k)^+$ based on the geometric average has a closed-form solution that is amenable to representation via a BSM-type formula.

To begin with, consider the case of discrete monitoring, for which

$$S_d^G(T;\sigma) \;=\; s\exp\biggl\{\frac{1}{m}\sum_{i=1}^{m}\bigl[(\mu-\tfrac{\sigma^2}{2})t_i + \sigma\mathcal{W}(t_i)\bigr]\biggr\}.$$

On the equally spaced time mesh $t_i = iT/m, i = 1,2,\ldots,m$, we have $\mathrm{Var}\bigl(\frac{\sigma}{m}\sum_{i=1}^m \mathcal{W}(t_i)\bigr)$ $= \frac{\sigma^2 T}{m^2}\sum_{i=1}^m\sum_{j=1}^m \min(i,j) \;=\; \iota(m)\sigma^2 T$, where $\iota(m) \equiv \frac{(2m+1)(m+1)}{6m^2}$ defines an "internal" volatility conversion factor. Letting $\kappa(m) \equiv \frac{m+1}{2m}$ be the "internal" time conversion factor, it follows that on the mesh $t_i = iT/m, i = 1,2,\ldots,m$, we have

$$S_d^G(T;\sigma) \;\sim\; s\exp\Bigl\{\mathrm{Nor}\bigl((\mu-\tfrac{\sigma^2}{2})\kappa(m)T, \iota(m)\sigma^2 T\bigr)\Bigr\}. \tag{20}$$

Now for fixed T and $m \to \infty$, it is clear that $\kappa(m) \to 1/2$ and $\iota(m) \to 1/3$, thus giving us the continuously monitord version

$$S_c^G(T;\sigma) \;\sim\; s\exp\Bigl\{\mathrm{Nor}\bigl((\mu-\tfrac{\sigma^2}{2})\tfrac{T}{2},\, \tfrac{\sigma^2 T}{3}\bigr)\Bigr\}. \tag{21}$$

Clearly, $S_c^G(T;\sigma)$ is lognormal, and it follows that we can directly apply the BSM formula to price a call on the geometric average. The BSM solution to the valuation of the continuously monitored geometric average option $C_c^G(\boldsymbol{\upsilon};\sigma) \equiv (S_c^G(T;\sigma) - k)^+$ is (see the appendix to this chapter)

$$c_c^G(\boldsymbol{\upsilon};\sigma) \;\equiv\; \mathrm{e}^{-rT}\mathrm{E}[C_c^G(\boldsymbol{\upsilon};\sigma)] \;=\; s\,\mathrm{e}^{-(r+\frac{\sigma^2}{6})\frac{T}{2}}\,\Phi(z_+^G) - k\,\mathrm{e}^{-rT}\,\Phi(z_-^G), \tag{22}$$

where

$$z_+^G \;\equiv\; \frac{\ell\mathrm{n}\bigl(\frac{s}{k}\bigr) + \Bigl(r+\frac{\sigma^2}{6}\Bigr)\frac{T}{2}}{\sigma\sqrt{\frac{T}{3}}} \qquad \text{and} \qquad z_-^G \;\equiv\; \frac{\ell\mathrm{n}\bigl(\frac{s}{k}\bigr) + \Bigl(r-\frac{\sigma^2}{2}\Bigr)\frac{T}{2}}{\sigma\sqrt{\frac{T}{3}}}. \tag{23}$$

By analogous reasoning and Equation (20), a valuation of the discretely monitored geometric average option is easily obtained.

2.3.3.5.2 Geometric Average with Unknown σ

By comparing the distributions of $S(t;\sigma)$ and $S_d^G(t;\sigma)$ from (1) and (20), and then carrying out the same manipulations as those leading to the c.d.f. of $S(t;\widehat{\sigma}_n)$ given in (5) of Lemma 1, we readily obtain for the discretely monitored case of $S_d^G(t;\widehat{\sigma}_n)$, $n \geq 2$, the c.d.f.

$$F_{S_d^G(t;\widehat{\sigma}_n)}(y) \quad = \quad \frac{n-1}{\sigma^2}\int_0^\infty \Phi\left(\frac{\ell\text{n}(\frac{y}{s}) - \left(\mu - \frac{w}{2}\right)\kappa(m)t}{\sqrt{\iota(m)wt}}\right) f_{\chi^2_{n-1}}\left(\tfrac{(n-1)w}{\sigma^2}\right)\mathrm{d}w, \quad y > 0.$$

Now for fixed T, since $\kappa(m) \to 1/2$ and $\iota(m) \to 1/3$, as $m \to \infty$, we find that the c.d.f. of $S_c^G(t;\widehat{\sigma}_n)$ in the continuously monitored case is

$$F_{S_c^G(t;\widehat{\sigma}_n)}(y) \quad = \quad \frac{n-1}{\sigma^2}\int_0^\infty \Phi\left(\frac{\ell\text{n}(\frac{y}{s}) - \left(\mu - \frac{w}{2}\right)\frac{t}{2}}{\sqrt{wt/3}}\right) f_{\chi^2_{n-1}}\left(\tfrac{(n-1)w}{\sigma^2}\right)\mathrm{d}w, \quad y > 0.$$

With appropriate substitution — analogous to (15) — for the discrete or continuous case, we can readily compute numerically

$$\begin{aligned} c_\star^G(\boldsymbol{v};\widehat{\sigma}_n) \quad &\equiv \quad \mathrm{e}^{-rT}\mathrm{E}[C_\star^G(\boldsymbol{v};\widehat{\sigma}_n)] \;\equiv\; \mathrm{e}^{-rT}\mathrm{E}[(S_\star^G(T;\sigma)-k)^+] \\ &= \quad \mathrm{e}^{-rT}\int_k^\infty \bar{F}_{S_\star^G(T;\widehat{\sigma}_n)}(y)\,\mathrm{d}y, \;\; \star = d, c. \end{aligned} \tag{24}$$

2.3.3.5.3 Arithmetic Average with Known σ

We now examine an Asian option based on the arithmetic average of GBM with known σ, i.e., $S_c^A(T;\sigma) \equiv \frac{1}{T}\int_0^T S(t;\sigma)\,\mathrm{d}t$ (continuously monitored) or $S_d^A(T;\sigma) \equiv \frac{1}{m}\sum_{i=1}^m S(t_i;\sigma)$ (discretely monitored) [55]. As these functionals of GBM lack the GBM representation, a closed-form BSM-type formula does not exist for this Asian; so in what follows, we will turn to simulation to price the call, $C_\star^A(\boldsymbol{v};\sigma) \equiv (S_\star^A(T;\sigma)-k)^+, \star = d, c$.

For convenience, we henceforth assume that m is sufficiently large so that $S_d^G(T;\sigma)$ and $S_d^A(T;\sigma)$ are approximations (to any specified tolerance) of their respective continuous versions $S_c^G(T;\sigma)$ and $S_c^A(T;\sigma)$, allowing us — for all practical purposes — to assume that we are dealing with the continuous case. To this end, dropping the indicator subscript of function type in Equation (24), we use Algorithm 1 to simulate ℓ independent replications of the sample path of the equity price, denoted by $(S_j(t;\sigma) : 0 \leq t \leq T)$ for $j = 1, 2, \ldots, \ell$. To illustrate for the discretely monitored case, we let $S_j^A(T;\sigma) \equiv \frac{1}{m}\sum_{i=1}^m S_j(t_i;\sigma)$ and

$C_j^A(\boldsymbol{v};\sigma) \equiv (S_j^A(T;\sigma) - k)^+$ for replication j. In order to estimate the price of the Asian arithmetic call, $c^A(\boldsymbol{v};\sigma) \equiv \mathrm{e}^{-rT}\mathrm{E}[C^A(\boldsymbol{v};\sigma)]$, we can use the *crude Monte Carlo* (MC) estimator $\hat{c}^A(\boldsymbol{v};\sigma) \equiv \frac{\mathrm{e}^{-rT}}{\ell}\sum_{j=1}^{\ell} C_j^A(\boldsymbol{v};\sigma)$, which averages the $C_j^A(\boldsymbol{v};\sigma)$'s over the ℓ replications and accounts for the time value of money.

Similarly, define $S_j^G(T;\sigma) \equiv \left(\prod_{i=1}^{m} S_j(t_i;\sigma)\right)^{1/m}$ and $C_j^G(\boldsymbol{v};\sigma) \equiv (S_j^G(T;\sigma) - k)^+$ for replications $j = 1, 2, \ldots, \ell$ of the Asian geometric call. Along the lines of the above discussion, the crude MC estimator for the price of the geometric call is given by $\hat{c}^G(\boldsymbol{v};\sigma) \equiv \frac{\mathrm{e}^{-rT}}{\ell}\sum_{j=1}^{\ell} C_j^G(\boldsymbol{v};\sigma)$ and is unbiased for $c^G(\boldsymbol{v};\sigma)$. We now use *common random numbers* (CRN) [23] to combine the crude MC estimators $\hat{c}^A(\boldsymbol{v};\sigma)$ and $\hat{c}^G(\boldsymbol{v};\sigma)$ to produce a *control variate* (CV) estimator for the arithmetic average Asian-type contingent claim,

$$\hat{\hat{c}}^A(\boldsymbol{v};\sigma) \;\equiv\; \hat{c}^A(\boldsymbol{v};\sigma) - \beta\big[\hat{c}^G(\boldsymbol{v};\sigma) - c^G(\boldsymbol{v};\sigma)\big], \tag{25}$$

where, for ease of exposition, we henceforth take $\beta = 1$ (see [49], which discusses the optimal selection of β). Both the crude MC estimator $\hat{c}^A(\boldsymbol{v};\sigma)$ and the CV estimator $\hat{\hat{c}}^A(\boldsymbol{v};\sigma)$ are clearly unbiased for $c^A(\boldsymbol{v};\sigma)$. However, since $C^A(\boldsymbol{v};\sigma)$ is highly correlated with $C^G(\boldsymbol{v};\sigma)$, the CV estimator is likely to have lower variance than the crude MC estimator [23].

2.3.3.5.4 Arithmetic Average with Unknown σ

Finally, we consider an Asian call $C^A(\boldsymbol{v};\widehat{\sigma}_n) \equiv (S^A(T;\widehat{\sigma}_n) - k)^+$, where $S^A(T;\widehat{\sigma}_n)$ is the arithmetic average of GBM with unknown σ over the time interval $[0, T]$. In the absence of a closed-form expression for $c^A(\boldsymbol{v};\widehat{\sigma}_n) \equiv \mathrm{e}^{-rT}\mathrm{E}[C^A(\boldsymbol{v};\widehat{\sigma}_n)]$, we again appeal to a CV estimator. Before doing so, let $C_j^A(\boldsymbol{v};\widehat{\sigma}_{n,j})$, $C_j^G(\boldsymbol{v};\widehat{\sigma}_{n,j})$, and $\widehat{\sigma}_{n,j}$, $j = 1, 2, \ldots, \ell$, denote independent replications of $C^A(\boldsymbol{v};\widehat{\sigma}_n)$, $C^G(\boldsymbol{v};\widehat{\sigma}_n)$, and $\widehat{\sigma}_n$, respectively. In our CV set up, the random variables $C_j^A(\boldsymbol{v};\widehat{\sigma}_{n,j})$ and $C_j^G(\boldsymbol{v};\widehat{\sigma}_{n,j})$ from replication j are calculated using CRN, i.e., using the same sample path $(S_j(t;\widehat{\sigma}_{n,j}) : 0 \le t \le T)$. Then our CV estimator for $c^A(\boldsymbol{v};\widehat{\sigma}_n)$ is given by

$$\hat{\hat{c}}^A(\boldsymbol{v};\widehat{\sigma}_n) \;\equiv\; \hat{c}^A(\boldsymbol{v};\widehat{\sigma}_n) - \big[\hat{c}^G(\boldsymbol{v};\widehat{\sigma}_n) - c^G(\boldsymbol{v};\widehat{\sigma}_n)\big], \tag{26}$$

where the crude MC estimators are $\hat{c}^A(\boldsymbol{v};\widehat{\sigma}_n) \equiv \frac{\mathrm{e}^{-rT}}{\ell}\sum_{j=1}^{\ell} C_j^A(\boldsymbol{v};\widehat{\sigma}_{n,j})$ and $\hat{c}^G(\boldsymbol{v};\widehat{\sigma}_n) \equiv \frac{\mathrm{e}^{-rT}}{\ell}\sum_{j=1}^{\ell} C_j^G(\boldsymbol{v};\widehat{\sigma}_{n,j})$, and $c^G(\boldsymbol{v};\widehat{\sigma}_n)$ is known from (24). (See [49] for additional motivation.)

Example 6 We value a variety of vanilla and Asian calls. Entries in Table 4 give, for all crude MC and CV (with $\beta = 1$) estimators, sample averages and standard errors based on $\ell = 10^5$ independent replications. The input parameters for the valuations are $T = 1/6$, $s = 10$, $r = 0.05$, and $\sigma = 1$, with the accompanying estimator $\widehat{\sigma}_n$ based on $n = 4$. We discretize the two-month ($T = 1/6$) time period by taking $m = 176$ equally spaced equity price observations (4 daily observations $\times$ 22 days $\times$ 2 months). The table gives results for strike prices $k = 8$, 9, 10, 11, and 12.

Table 4: Exact and Estimated Call Values with $\boldsymbol{v} = (s, k, T) = (10, k, 1/6)$, $r = 0.05$, $\sigma = 1$, $n = 4$, and $m = 176$

k	8	9	10	11	12
$c(\boldsymbol{v};\sigma)$	2.706	2.126	1.653	1.274	0.977
$c^G(\boldsymbol{v};\sigma)$	2.093	1.398	0.879	0.523	0.297
$\hat{c}^A(\boldsymbol{v};\sigma)$	2.201 (0.007)	1.490 (0.006)	0.956 (0.005)	0.586 (0.004)	0.346 (0.003)
$\hat{\hat{c}}^A(\boldsymbol{v};\sigma)$	2.197 (0.0004)	1.486 (0.0004)	0.952 (0.0004)	0.583 (0.0005)	0.344 (0.0004)
$c(\boldsymbol{v};\widehat{\sigma}_4)$	2.651	2.023	1.523	1.151	0.882
$c^G(\boldsymbol{v};\widehat{\sigma}_4)$	2.098	1.357	0.804	0.465	0.275
$\hat{c}^A(\boldsymbol{v};\widehat{\sigma}_4)$	2.206 (0.007)	1.451 (0.006)	0.885 (0.006)	0.533 (0.005)	0.329 (0.004)
$\hat{\hat{c}}^A(\boldsymbol{v};\widehat{\sigma}_4)$	2.201 (0.0007)	1.445 (0.0007)	0.878 (0.0007)	0.527 (0.0007)	0.326 (0.0007)

The $c(\boldsymbol{v};\sigma)$ and $c(\boldsymbol{v};\widehat{\sigma}_4)$ rows of Table 4 respectively provide the exact BSM pre- and post-estimation vanilla call values. Similarly, the $c^G(\boldsymbol{v};\sigma)$ and $c^G(\boldsymbol{v};\widehat{\sigma}_4)$ rows give the exact pre- and post-estimation geometric average call values. The $\hat{c}^A(\boldsymbol{v};\sigma)$ and $\hat{c}^A(\boldsymbol{v};\widehat{\sigma}_4)$ rows give estimated pre- and post-estimation arithmetic average call values obtained by crude MC, with standard errors in parentheses. Finally, the $\hat{\hat{c}}^A(\boldsymbol{v};\sigma)$ and $\hat{\hat{c}}^A(\boldsymbol{v};\widehat{\sigma}_4)$ rows give estimated pre- and post-estimation arithmetic average call values obtained using the CV estimators in Equations (25) and (26), again with standard errors in parentheses.

We note that:

- Not unexpectedly due to the "averaging" of the underlying, vanilla call values are more expensive than "average" call values.

- The results on Asian call values conform with the geometric-arithmetic average inequality [10], i.e., the geometric average is a lower bound for the arithmetic average.

- The CV arithmetic average estimators have almost the same means — with substantially lower standard errors — than their crude MC counterparts.
- Achieving a good continuous approximation of the discrete monitoring case typically requires the integer m to be "tuned," i.e., an appropriate choice of m depends on the parameters in the option contract.

2.3.3.6 Vanilla Greeks

The "Greeks" are a set of sensitivities of a financial instrument to unanticipated changes in the underlying parameters, and are typically used in devising and monitoring hedging strategies [55]. For example, in the case of a vanilla BSM call, the most-frequently used Greek is $\delta(\boldsymbol{v};\sigma) \equiv \frac{\partial c(\boldsymbol{v};\sigma)}{\partial s}$, which gives the sensitivity of the option value to a change in the observed price of the underlying. Under ideal circumstances, i.e., when all the assumptions of the BSM world are satisfied, this sensitivity indicates how many additional units of the underlying one needs to go short or long in order to balance out in value a portfolio consisting of the call, the underlying stock, and a money market account. We consider four versions of delta: the noted BSM; $\delta(\boldsymbol{v};\sigma)$, our post-estimation $\delta(\boldsymbol{v};\widehat{\sigma}_n)$; the LUQ delta $\delta^*(\boldsymbol{v};\widehat{\sigma}_n) \equiv \frac{n-1}{\sigma^2}\Phi\big(z_+(\Xi)\big)$, where $\Xi \sim \frac{\sigma^2}{n-1}\chi^2_{n-1}$; and a plug-in delta $\widehat{\delta}$ that uses the current estimate of volatility. For given instantiating parameters, $\delta(\boldsymbol{v};\sigma)$, $\delta(\boldsymbol{v};\widehat{\sigma}_n)$, and $\delta^*(\boldsymbol{v};\widehat{\sigma}_n)$ are numbers, i.e., expectations, whereas $\widehat{\delta}$ is a random variable. To each of the initial four deltas there corresponds a gamma, $\gamma(\boldsymbol{v};\sigma)$ (defined in Table 6), $\gamma(\boldsymbol{v};\widehat{\sigma}_n)$, $\gamma^*(\boldsymbol{v};\widehat{\sigma}_n)$, and $\widehat{\gamma}$. Useful sensitivities are the matching vegas $\vartheta(\boldsymbol{v};\sigma)$ (defined in Table 6), $\vartheta(\boldsymbol{v};\widehat{\sigma}_n)$, $\vartheta^*(\boldsymbol{v};\widehat{\sigma}_n)$, and $\widehat{\vartheta}$. It turns out that for the vanilla call, the LUQ delta and gamma ($\gamma^*(\boldsymbol{v};\widehat{\sigma}_n) \equiv \frac{n-1}{\sigma^2}\frac{1}{s\sigma\sqrt{\tau}}\,\phi\big(z_+(\Xi)\big)$, $\Xi \sim \frac{\sigma^2}{n-1}\chi^2_{n-1}$) are the same as those in our post-estimation case. However, as we show below, the vegas differ.

Table 5: Classic BSM Greeks

Greek	defined	call formula	put formula
δ	$\frac{\partial c(\boldsymbol{v};\sigma)}{\partial s}$	$\Phi(z_+)$	$\Phi(z_+) - 1$
γ	$\frac{\partial^2 c(\boldsymbol{v};\sigma)}{\partial s^2}$	$\frac{1}{s\sigma\sqrt{\tau}}\,\phi(z_+)$	$\frac{1}{s\sigma\sqrt{\tau}}\,\phi(z_+)$
ϑ	$\frac{\partial c(\boldsymbol{v};\sigma)}{\partial \sigma}$	$s\phi(z_+)\sqrt{\tau}$	$s\phi(z_+)\sqrt{\tau}$

In the classic BSM formulation, when the stock is non-dividend paying and σ is known, we have that $\delta(\boldsymbol{v};\sigma) = \Phi(z_+)$ [55]. The corresponding post-estimation delta is the positive function

$$\begin{aligned}\delta(\boldsymbol{v};\widehat{\sigma}_n) &= \frac{\partial c(\boldsymbol{v};\widehat{\sigma}_n)}{\partial s} \\ &= \frac{\mathrm{e}^{-rT}(n-1)}{s\sigma^2}\int_k^\infty\int_0^\infty \frac{1}{\sqrt{wT}}\,\phi\left(\frac{\ell\mathrm{n}(\frac{y}{s}) - \left(\mu - \frac{w}{2}\right)T}{\sqrt{wT}}\right) f_{\chi^2_{n-1}}\left(\tfrac{(n-1)w}{\sigma^2}\right)\mathrm{d}w\,\mathrm{d}y.\end{aligned}$$

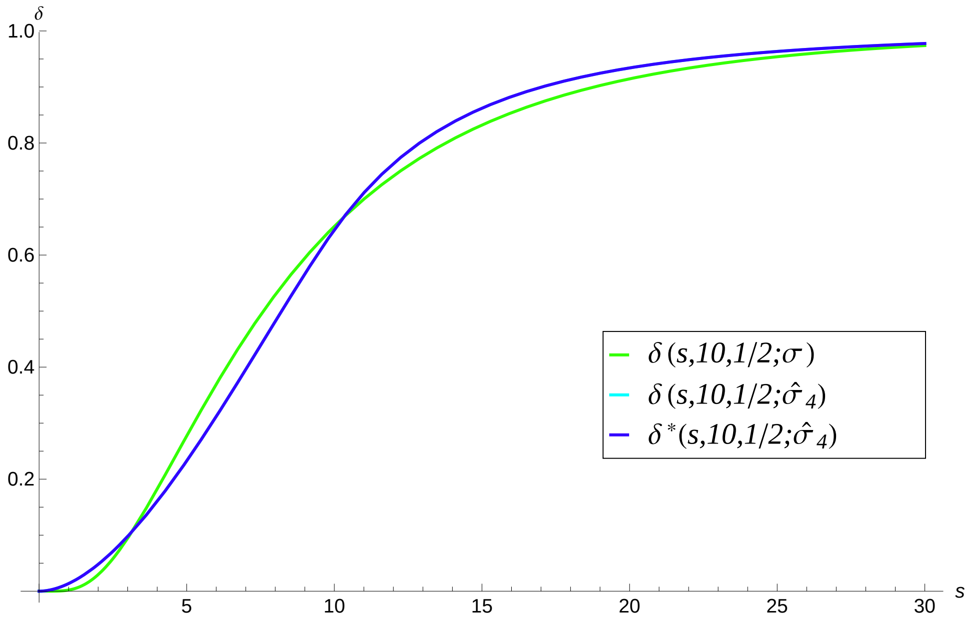
Similarly, we have a post-estimation gamma and vega. Using the change of variables $u = \frac{n-1}{\sigma^2}w$ into Equation (5), the post-estimation formula for vega is

$$\begin{aligned}\vartheta(\boldsymbol{v};\widehat{\sigma}_n) &= \frac{\partial c(\boldsymbol{v};\widehat{\sigma}_n)}{\partial \sigma} \\ &= \mathrm{e}^{-rT}\int_k^\infty\int_0^\infty \frac{\left(\ell\mathrm{n}(\frac{y}{s}) - \left(\mu + \frac{\sigma^2 u}{2(n-1)}\right)T\right)\sigma u T}{(n-1)^{5/2}\sigma^3(uT)^{3/2}}\phi\left(z(u)\right) f_{\chi^2_{n-1}}\left(u\right)\mathrm{d}u\,\mathrm{d}y > 0,\end{aligned}$$

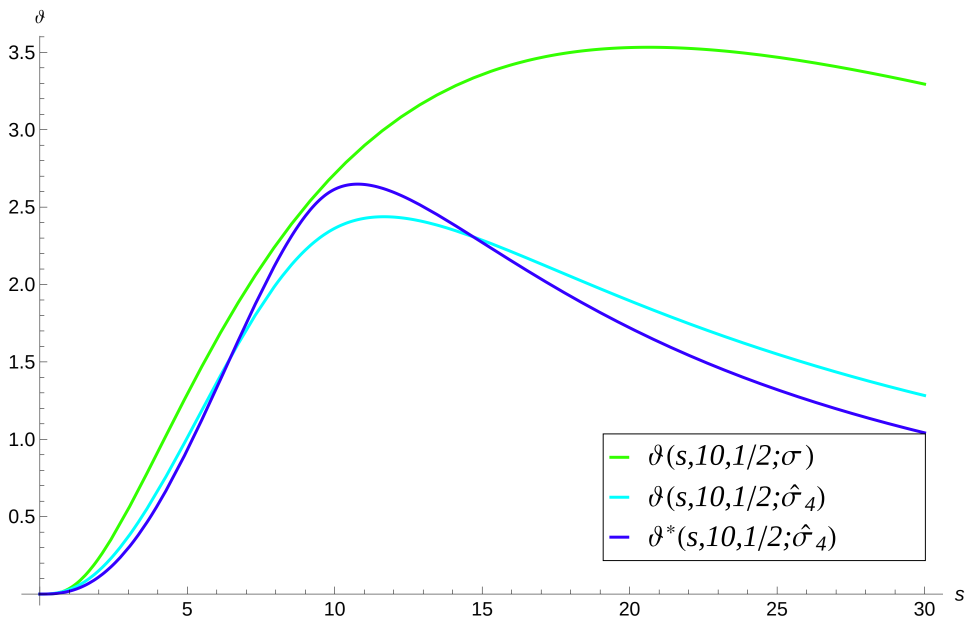
where $z(u) \equiv \frac{\ell\mathrm{n}(\frac{y}{s}) - \left(\mu - \frac{\sigma^2 u}{2(n-1)}\right)T}{\sqrt{\frac{\sigma^2 uT}{n-1}}}$.

Example 7 Figure 7(a) compares the BSM $\delta(\boldsymbol{v};\sigma)$, our post-estimation $\delta(\boldsymbol{v};\widehat{\sigma}_n)$, and the LUQ version $\delta^*(\boldsymbol{v};\widehat{\sigma}_n)$. At-the-money (ATM) is $s = k = 10$, and the other operating parameter choices are set at $T = 1$, $r = 0.05$, and $\sigma = 1$, with $n = 4$. As we see from the figure, $\delta(\boldsymbol{v};\widehat{\sigma}_n)$ and $\delta^*(\boldsymbol{v};\widehat{\sigma}_n)$ are coincident and cross the BSM $\delta(\boldsymbol{v};\sigma)$ several times, eventually converging to the same values for large s. The intuition for the serendipitous match in our $\delta(\boldsymbol{v};\widehat{\sigma}_n)$ and that of the LUQ $\delta^*(\boldsymbol{v};\widehat{\sigma}_n)$ is due to the cancellation that occurs in the differentiation, with respect to s, of the BSM vanilla call formula — the first term of the BSM call formula (up to a factor of proportionality s) is essentially reproduced as the delta. One might, at this point, claim that there is no difference between the results obtained via the post-estimation c.d.f and that obtained by invoking the LUQ. Figure 7(b) shows that such a claim is false. Here we see that the vegas differ substantially. The LUQ vega $\vartheta^*(\boldsymbol{v};\widehat{\sigma}_n)$, when compared to ours, better represents BSM for all s values near ATM. But then we have been arguing in this paper that our model is a better reflection of economic reality. We present the "other" Greeks, e.g., vega, etc., and methods to calculate them in the next chapter.

In the final comparison, we examine the implications of naive hedging in which one merely plugs the estimator $\widehat{\sigma}_n$ into a particular BSM Greek. Proposition 1 gives the c.d.f.



(a) BSM $\delta(\boldsymbol{v};\sigma)$ vs. $\delta(\boldsymbol{v};\widehat{\sigma}_4)$ and $\delta^*(\boldsymbol{v};\widehat{\sigma}_4)$: $k = 10$; $T = 1/2$; $r = 0.05$; $\sigma = 1$



(b) BSM $\vartheta(\boldsymbol{v};\sigma)$ vs. $\vartheta(\boldsymbol{v};\widehat{\sigma}_4)$ and $\vartheta^*(\boldsymbol{v};\widehat{\sigma}_4)$: $k = 10$; $T = 1/2$; $r = 0.05$; $\sigma = 1$

Figure 7: Three δ's and three accompanying ϑ's

of such an estimator for δ, i.e., $\widehat{\delta} \equiv \Phi(\widehat{z}_+)$, where from Equation (3), $\widehat{z}_+$ is z_+ with $\widehat{\sigma}_n$ in place of σ. The proposition also indicates that "delta hedging" of the naive type is biased and may exhibit substantial variability.

Proposition 1 The c.d.f. of $\widehat{\delta}$ is

$$F_{\widehat{\delta}}(d) \;\equiv\; \Pr\big(\Phi(\widehat{z}_+) \leq d\big) \;=\; \big[F_{\chi^2_{n-1}}(e_+(d)) - F_{\chi^2_{n-1}}(e_-(d))\big] I_{\{q(d)>0\}},$$

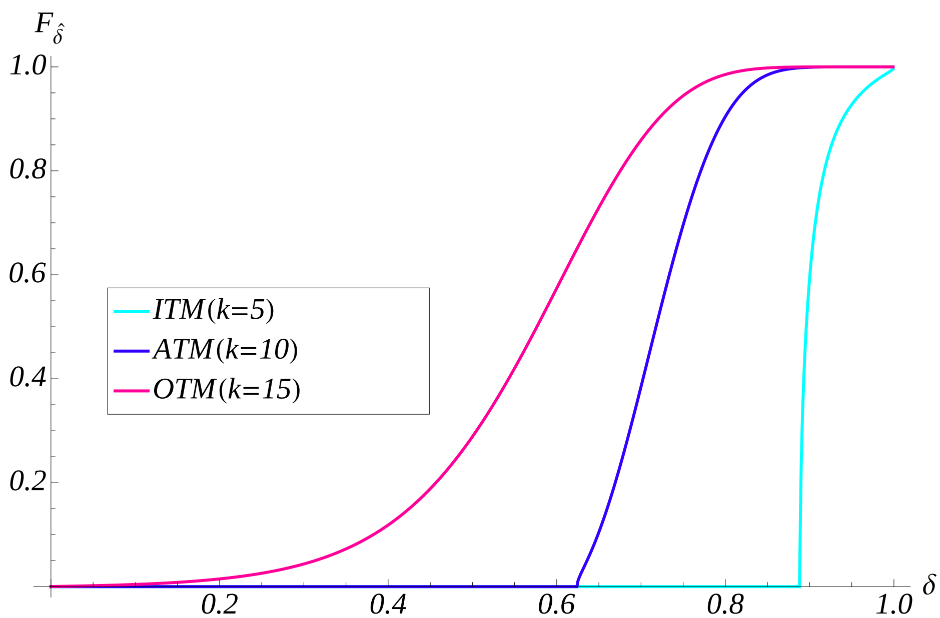
where $F_{\chi^2_{n-1}}(\cdot)$ denotes the χ^2_{n-1} c.d.f., $e_\pm(d) \equiv (n-1)\big[\big(b(d) \pm \sqrt{q(d)}\,\big)^+\big]^2/\sigma^2$, $b(d) \equiv \Phi^{-1}(d)/\sqrt{T}$, $q(d) \equiv b^2(d) - 2a/\sqrt{T}$ for $0 \leq d \leq 1$, and $a \equiv \big[\ell\mathrm{n}(\frac{s}{k}) + \mu T\big]/\sqrt{T}$.

Example 8 Figure 8 concerns the c.d.f. of the estimator $\widehat{\delta}$ as given in Proposition 1. Figure 8(a) assumes $n = 4$ and depicts three c.d.f.'s with $s = 10$, $T = 1$, $\sigma = 1$, and $r = 0.05$: namely, ITM ($k = 5$), ATM ($k = 10$), and OTM ($k = 15$). As we progress further ITM, the c.d.f. approaches a point probability. Figure 8(b) compares the ATM cases with $n = 4$ and 1000 — the latter effectively being classic BSM. Note that classic BSM, as expected for known σ, produces a c.d.f. that is a point probability.

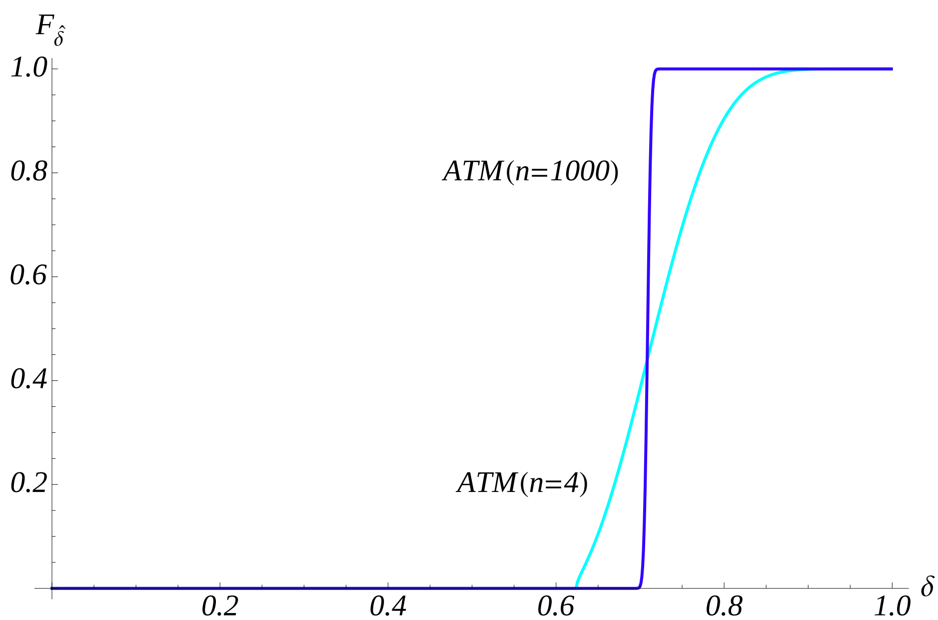
2.4 Conclusions

Our concern is with the assessment, or lack thereof, of estimation risk in financial modeling. Surprisingly, great attention is paid by practitioners at financial institutions to a few basis points when it comes to gauging a "model's fit," yet less stress is placed on formally incorporating the uncertainty attached to the fundamental parameters of a model and to what the consequences of that uncertainty are for a firm's present and future bottom line. Model fit may be improved by adding parameters, but at the cost of increased out-of-sample variability. For purposes of prediction, neglecting the variability of the available data used in the calibration of a model is analogous to failing to incorporate for friction or wind effects when calculating the trajectory of a missile — it can be consequential. We showed that simply applying the expedient of the LUQ often leads to substantial differences in valuation and hedging actions. Additional results dealing with these issues are developed in the next Chapter 3.

For illustrative purposes, we intentionally used in our analysis a specific fundamental process — constant-coefficients GBM — whose properties are well known. Extensions we



(a) $n = 4$; $k = 5,\ 10,\ 15$; $s = 10$; $\sigma = 1$; $T = 1$; and $r = 0.05$



(b) $n = 4,\ 1000$; $s = k = 10$; $T = 1$; $\sigma = 1$; and $r = 0.05$

Figure 8: A cornucopia of c.d.f.'s for $\widehat{\delta}$

are pursuing include more-general Lévy processes, the development of a comprehensive Greeks paradigm, and the application of our methods — including change point analysis of volatility — to the valuation and hedging of large portfolios consisting of assorted options on different underlying equities, e.g., straddles, collars, etc. To each underlying equity there corresponds a particular estimation risk which in turn, based on our analysis, transfers in a predictable way to the contingent claims that are written on it.

An open subject for future research is the development of the above "single" agent view, but within the context of a general equilibrium model where the parameter n, indicative of a market's information content, achieves an equilibrium value. We conjecture that such a set-up will generate an endogenous "fake" smile effect. If agents in different strike and expiry segments of an option market perceive — subject to the same functional estimator — differing values of volatility, then a fake smile can be the outcome when estimates for volatility are plugged into BSM. Why should the use of differing data frequencies or estimation segments give different values for volatility? This phenomenon is plainly due to the fact that volatility does depend on the estimation horizon — it does change unpredictably, and at those change point times is more in line with Knight's definition of uncertainty. Over interim periods σ is a constant amenable to the analysis we have outlined.

2.5 Appendix

This appendix is devoted to proving the various new results we introduce in the body of the paper.

Proof of Lemma 1 We have

$$\begin{aligned}
F_{S(t;\widehat{\sigma})}(y) &= \int_0^\infty \Pr\left(S(t;\widehat{\sigma}) \le y \,|\, \widehat{\sigma}^2 = w\right) f_{\widehat{\sigma}^2}(w)\,\mathrm{d}w \\
&= \int_0^\infty \Pr\left(\ell\mathrm{n}(s) + \left(\mu - \frac{\widehat{\sigma}^2}{2}\right) t + \widehat{\sigma}\,\mathcal{W}(t) \le \ell\mathrm{n}(y) \,\middle|\, \widehat{\sigma}^2 = w\right) f_{\widehat{\sigma}^2}(w)\,\mathrm{d}w \\
&= \int_0^\infty \Pr\left(\ell\mathrm{n}(s) + \left(\mu - \frac{w}{2}\right) t + \sqrt{w}\,\mathcal{W}(t) \le \ell\mathrm{n}(y)\right) f_{\widehat{\sigma}^2}(w)\,\mathrm{d}w,
\end{aligned}$$

which follows since $\widehat{\sigma}^2$ is independent of $\mathcal{W}(t)$ by assumption. □

Proof of Corollary 1 The expression for $F_{S(t;\widehat{\sigma}_n)}(y)$ follows easily from Lemma 1. In addition, we have

$$\begin{aligned} f_{S(t;\widehat{\sigma}_n)}(y) &= \frac{n-1}{y\sqrt{t}\,\sigma^2}\int_0^\infty \phi\left(\frac{a_0(y)+\frac{wt}{2}}{\sqrt{wt}}\right)\frac{1}{\sqrt{w}}\,f_{\chi^2_{n-1}}\left(\tfrac{(n-1)w}{\sigma^2}\right)\mathrm{d}w \\ &= \frac{n-1}{y\sqrt{t}\,\sigma^2}\int_0^\infty \frac{1}{\sqrt{2\pi}}\,\exp\left\{-\left(\frac{a_0^2(y)}{2wt}+\frac{a_0(y)}{2}+\frac{wt}{8}\right)\right\}\frac{1}{\sqrt{w}} \\ &\qquad \times\frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}\left(\frac{(n-1)w}{\sigma^2}\right)^{\frac{n-3}{2}}\exp\left\{-\frac{(n-1)w}{2\sigma^2}\right\}\mathrm{d}w, \end{aligned}$$

from which we obtain the desired result. □

Proof of Corollary 2 Let $\zeta(w)\equiv \ell\mathrm{n}(s)+(\mu-\frac{w}{2})t$. Starting at Equation (6), we find that

$$\begin{aligned} \mathrm{E}[S^j(t;\widehat{\sigma})] &= \int_0^\infty y^j f_{S(t;\widehat{\sigma})}(y)\,\mathrm{d}y \\ &= \int_0^\infty y^j\int_0^\infty \frac{1}{y\sqrt{wt}}\,\phi\left(\frac{\ell\mathrm{n}(\frac{y}{s})-\left(\mu-\frac{w}{2}\right)t}{\sqrt{wt}}\right)f_{\widehat{\sigma}^2}(w)\,\mathrm{d}w\,\mathrm{d}y \\ &= \int_0^\infty f_{\widehat{\sigma}^2}(w)\int_0^\infty y^{j-1}\,\frac{1}{\sqrt{wt}}\,\phi\left(\frac{\ell\mathrm{n}(y)-\zeta(w)}{\sqrt{wt}}\right)\mathrm{d}y\,\mathrm{d}w \\ &= \int_0^\infty f_{\widehat{\sigma}^2}(w)\int_0^\infty y^{j-1}\,\frac{1}{\sqrt{2\pi wt}}\,\exp\left[-\frac{(\ell\mathrm{n}(y)-\zeta(w))^2}{2wt}\right]\mathrm{d}y\,\mathrm{d}w \\ &= \int_0^\infty f_{\widehat{\sigma}^2}(w)\exp\left[\zeta(w)j+\frac{wtj^2}{2}\right]\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi wt}}\,\exp\left[-\frac{(z-(\zeta(w)+wtj))^2}{2wt}\right]\mathrm{d}z\,\mathrm{d}w \\ &= s^j\mathrm{e}^{\mu tj}\int_0^\infty f_{\widehat{\sigma}^2}(w)\exp\left[\frac{t}{2}(j^2-j)w\right]\mathrm{d}w, \end{aligned}$$

where the penultimate step follows after setting $y=\mathrm{e}^z$ and completing the square in the exponent; and the final step follows after noting that the interior integrand is a normal p.d.f. and simplifying. □

Proof of Proposition 1 By definition, we have

$$\begin{aligned} F_{\widehat{\delta}}(d) &= \Pr\big(\Phi(\widehat{z}_+)\le d\big) \;=\; \Pr\left(\ell\mathrm{n}\left(\tfrac{s}{k}\right)+\left(\mu+\frac{\widehat{\sigma}_n^2}{2}\right)T\le \Phi^{-1}(d)\,\widehat{\sigma}_n\sqrt{T}\right) \\ &= \Pr\left(\widehat{\sigma}_n^2-2b(d)\,\widehat{\sigma}_n+\frac{2a}{\sqrt{T}}\le 0\right) \;=\; \Pr\left(\big(\widehat{\sigma}_n^2-b(d)\big)^2\le b^2(d)-\frac{2a}{\sqrt{T}}\right) \\ &= \Pr\big(-\sqrt{q(d)}\le\widehat{\sigma}_n-b(d)\le\sqrt{q(d)}\,\big)I_{\{q(d)>0\}} \\ &= \Pr\Big(\big(b(d)-\sqrt{q(d)}\,\big)^+\le\widehat{\sigma}_n\le\big(b(d)+\sqrt{q(d)}\,\big)^+\Big)I_{\{q(d)>0\}} \\ &= \Pr\left(\frac{(n-1)\big[\big(b(d)-\sqrt{q(d)}\,\big)^+\big]^2}{\sigma^2}\le\frac{(n-1)\widehat{\sigma}_n^2}{\sigma^2}\le\frac{(n-1)\big[\big(b(d)+\sqrt{q(d)}\,\big)^+\big]^2}{\sigma^2}\right)I_{\{q(d)>0\}}. \quad \Box \end{aligned}$$

Proof of Barrier Equation (17) First observe that for standard BM, the individual process $U(t)\equiv\max\{\mathcal{W}(s),\,0<s\le t\}$, is not Markov, whereas the bivariate process $((\mathcal{W}(t),U(t)),t\ge 0)$ is Markov. This follows from the fact that the transition probability for $(\mathcal{W}(t),U(t))$ is fully characterized by knowledge of the present location of the state. Yet on the other hand, the transition probability of $U(t)$ alone not only depends on knowledge of what the running maximum is currently, but also on the current point on the Brownian path $(\mathcal{W}(t),t\ge 0)$. It follows that to obtain the probability of the event $\{U(t)\ge y\}$, we need the

joint c.d.f. $F_{\mathcal{W}(t),U(t)}(x,y)$, $x<y, y>0$. With the aid of the reflection principle [55], the c.d.f. and p.d.f. are

$$\begin{aligned} F_{\mathcal{W}(t),U(t)}(x,y) &= \Pr(\mathcal{W}(t)\le x, U(t)\le y) \\ &= \Pr(\mathcal{W}(t)\le x) - \Pr(\mathcal{W}(t)\le x, U(t)\ge y) \\ &= \Phi\Big(\frac{x}{\sqrt{t}}\Big) - \Phi\Big(\frac{x-2y}{\sqrt{t}}\Big), \quad x<y,\ y>0, \end{aligned} \tag{27}$$

and

$$f_{\mathcal{W}(t),U(t)}(x,y) = \sqrt{\frac{2}{\pi t^3}}\,(2y-x)\mathrm{e}^{-\frac{(2y-x)^2}{2t}}, \quad x<y,\ y>0. \tag{28}$$

Evidently, for any fixed $t>0$, the first passage time $\tau \equiv \min\{t>0 : \mathcal{W}(t)=B\}$ and the maximum $U(t)$ are related by the events $\{\tau\le t\} = \{U(t)\ge B\}$. So letting $(x,y)\to(b,b)$ in Equation (27), we obtain the c.d.f. and p.d.f. of τ,

$$F_\tau(b) = 2\Phi\Big(\frac{b}{\sqrt{t}}\Big) - 1, \quad b>0,$$

and

$$f_\tau(b) = \sqrt{\frac{2}{\pi t}}\,\mathrm{e}^{-\frac{b^2}{2t}}, \quad b>0,$$

which is the half-normal p.d.f. (typically associated with the random variable $|\mathcal{W}(t)|$).

We now follow the above steps, amended where necessary, to obtain comparable results for GBM. Our notation for the stopping time associated with GBM is $\theta \equiv \min\{t>0 : S(t;\sigma)=B\} = \min\{t>0 : \mathcal{W}(t) = \beta - \lambda t\}$, with $\beta \equiv \frac{1}{\sigma}\ell\mathrm{n}(\frac{B}{s})$ and $\lambda \equiv \frac{1}{\sigma}(r-\frac{\sigma^2}{2})$. Aside from sets of measure zero, it is true that

$$\{S(t;\sigma)\ge B\} \quad\Longleftrightarrow\quad \{\lambda t + \mathcal{W}(t) \ge \tfrac{1}{\sigma}\ell\mathrm{n}\big(\tfrac{B}{s}\big)\}.$$

Since the function $\ell\mathrm{n}(\cdot)$ is strictly increasing, it follows that

$$\Pr(\theta\le T) \;=\; \Pr(M(T;\sigma)\ge B) \;=\; \Pr(\max{}_{t\le T}\{\lambda t + \mathcal{W}(t)\} \ge \beta).$$

Therefore, we conclude that Equation (17) is a probability statement concerning a standard Brownian motion *with drift* λt, crossing the barrier β.

A second change of measure, implemented below (Equation (29), third equality) will induce the Brownian motion $\mathcal{B}(t) = \lambda t + \mathcal{W}(t)$ to be driftless. Specifically, the Cameron–Martin–Girsanov Theorem [35] relates probability measures P and Q through $\mathrm{dQ} = \exp\{-\lambda\mathcal{W}(T) - \frac{\lambda^2 T}{2}\}\,\mathrm{dP}$. To this end, with $A<B$, define the new composite parameter $\alpha \equiv \frac{1}{\sigma}\ell\mathrm{n}(\frac{A}{s})$, and use Equation (28) to obtain via the "tilting" of measure Q to measure P,

$$\begin{aligned} F_{S(t;\sigma),M(T;\sigma)}(A,B) &= \Pr(S(t;\sigma)\le A, M(T;\sigma)\le B) \\ &= \Pr\Big(\lambda t + \mathcal{W}(t)\le\alpha,\ \max{}_{t\le T}\{\lambda t + \mathcal{W}(t)\}\le\beta\Big) \\ &= \mathrm{E}^{\mathrm{Q}}\Big[\frac{\mathrm{dP}}{\mathrm{dQ}} I_{\mathcal{B}(t)\le\alpha,\ \max_{t\le T}\mathcal{B}(t)\le\beta}\Big] \\ &= \Pr\Big(\mathcal{B}(t)\le\alpha,\ \max{}_{t\le T}\mathcal{B}(t)\le\beta\Big) \\ &= \sqrt{\frac{2}{\pi t^3}}\int\!\!\int_{\mathcal{S}} \mathrm{e}^{\lambda x - \frac{\lambda^2 T}{2}}(2y-x)\mathrm{e}^{-\frac{(2y-x)^2}{2T}}\,\mathrm{d}y\,\mathrm{d}x, \end{aligned} \tag{29}$$

where $\mathcal{S} \equiv \{(x,y) : -\infty < x < \alpha,\ x\le y\le\beta,\ \alpha<\beta\}$ is a convex set.

In Equation (29), requiring $A \to B$ forces $\alpha \to \beta$, and so $\mathcal{S} \to \mathcal{S}^* \equiv \{(x,y) : -\infty < x \le y \le \beta\} = \{(x,y) : -\infty < x < 0,\, 0 \le y \le \beta\} \cup \{(x,y) : 0 < x < \beta,\, x \le y \le \beta\}$. Integrating over $\mathcal{S}^*$, and substituting for β and λ, we obtain Equation (17). For the details of the integration over $\mathcal{S}^*$, so as to obtain Equation (17), consult [55] or use an algebra manipulator such as Mathematica. Lastly when dealing with the post-estimation case, the next to last equality in Equation (29) is where Lemma 1 is invoked. $\square$

Proof of Forward Start Equation (19) It is convenient to introduce a filtration [35] generated by the Brownian motion, i.e., $\mathcal{F}_0 \subset \mathcal{F}_{\bar{t}} \subset \mathcal{F}_t$, for any $\bar{t} < t$. Using Equation (18), we characterize the event

$$\psi \equiv \{S(T) > xS(\bar{T})\} = \left\{\xi > \frac{\ell\text{n}(x) - (r - \frac{\sigma^2}{2})(T - \bar{T})}{\sigma(T - \bar{T})^{1/2}}\right\},$$

where $\xi \sim \text{Nor}(0,1)$. With the replacements mentioned following Equation (19), we first calculate

$$\text{E}[S(T;\sigma)I_\psi |\, \mathcal{F}_{\bar{T}}] \;=\; S(\bar{T};\sigma)\text{E}[I_\psi |\, \mathcal{F}_{\bar{T}}] \;=\; S(\bar{T};\sigma)\, \text{e}^{r(T-\bar{T})}\Phi(z_+).$$

The initial equality in the above equation follows from the fact that the underlying process is adapted. Now $\text{E}[\text{E}[S(T;\sigma)I_\psi |\, \mathcal{F}_{\bar{T}}]|\, \mathcal{F}_0] = \text{E}[S(\bar{T};\sigma)|\, \mathcal{F}_0]\, \text{e}^{r(T-\bar{T})}\Phi(z_+) = s\, \text{e}^{rT}\Phi(z_+)$, where the last equality follows via risk-neutrality. Next we calculate

$$\text{E}[S(\bar{T};\sigma)I_\psi |\, \mathcal{F}_{\bar{T}}] \;=\; S(\bar{T};\sigma)\text{E}[I_\psi |\, \mathcal{F}_{\bar{T}}] \;=\; S(\bar{T};\sigma)\Phi(z_-),$$

but also $\text{E}[\text{E}[S(\bar{T};\sigma)I_\psi |\, \mathcal{F}_{\bar{T}}]|\, \mathcal{F}_0] = \text{E}[S(\bar{T};\sigma)|\, \mathcal{F}_0]\, \Phi(z_-) = s\, \text{e}^{r\bar{T}}\Phi(z_-)$. Finally, add the two components. $\square$

Proof of Geometric Asian BSM (22) Consider the ITM event, $\mathcal{E} \equiv \{S^G(T;\sigma) \ge k\} = \{\xi \ge \frac{1}{\sigma\sqrt{T/3}}\left[\ell\text{n}\left(\frac{k}{s}\right) - (r - \frac{\sigma^2}{2})\frac{T}{2}\right]\}$, where $\xi \sim \text{Nor}(0,1)$. By direct computation,

$$\begin{aligned}
\text{e}^{-rT}\text{E}[C^G(\boldsymbol{v};\sigma)] &= \text{e}^{-rT}\int_0^\infty (y-k)^+ f_{S^G(T;\sigma)}(y)\,\text{d}y \\
&= s\,\text{e}^{-(r+\frac{\sigma^2}{6})\frac{T}{2}}\int_{\mathcal{E}} \text{e}^{\sigma\sqrt{\frac{T}{3}}\xi}\phi(\xi)\,\text{d}\xi - \text{e}^{-rT}k\int_{\mathcal{E}}\phi(\xi)\,\text{d}\xi,
\end{aligned}$$

and Equations (22) and (23) follow. $\square$

Control Variate Approach to Arithmetic Average Option The CV approach aims to reduce the variance of certain point estimators. In what follows, keeping track of the discount factor e^{-rT} is just a matter of bookkeeping, and so without loss of generality we set $r = 0$. The three keys to the CV technique are:

1. Designate as a CV a random variable that has *high correlation*, ideally for all values in parameter space, with the random variable whose mean we wish to estimate — in our case, $C^A(\boldsymbol{v};\sigma)$ and $C^G(\boldsymbol{v};\sigma)$ are positively and highly correlated for all parameter values, and we desire to estimate $c^A(\boldsymbol{v};\sigma)$.

2. The CV *has a known* population expected value — in our case, via BSM and Equation (21) we can calculate the expected value $c^G(\boldsymbol{\upsilon};\sigma)$.

3. For variance reduction purposes, when jointly generating representatives of $C^A(\boldsymbol{\upsilon};\sigma)$ and $C^G(\boldsymbol{\upsilon};\sigma)$ and the post-estimation analogues, utilize *common random numbers* [23] to simulate $S^A(T;\sigma)$, $S^G(T;\sigma)$, $S^A(T;\widehat{\sigma}_n)$, and $S^G(T;\widehat{\sigma}_n)$.

Given the first two requirements above, since we are designing the simulation having a run length m, we posit that the two processes are related via an ordinary least squares regression,

$$\mathbf{C}_A^{\alpha,\beta} \equiv \alpha\mathbf{C}_A - \beta(\mathbf{C}_G - \mathbf{1}c_G) + \boldsymbol{\epsilon},$$

where $(\mathbf{C}_A, \mathbf{C}_G)$ are jointly generated vectors of claim values, c_G is of course a scalar with $\mathbf{1}$ an m vector of 1's, and $\boldsymbol{\epsilon}$ is an i.i.d. vector of error terms; see [21]. Taking expectations we get $\mathrm{E}[\mathbf{C}_A^{\alpha,\beta}] = \alpha\mathrm{E}[\mathbf{C}_A] - \beta\mathrm{E}[\mathbf{C}_G - \mathbf{1}c_G] + \mathrm{E}[\boldsymbol{\epsilon}]$, which in turn lets us set $\alpha = 1$. We therefore consider, with $\beta > 0$,

$$\mathbf{C}_A^{\beta} \;\equiv\; \mathbf{C}_A - \beta(\mathbf{C}_G - \mathbf{1}c_G) + \boldsymbol{\epsilon}, \tag{30}$$

such that $\mathrm{E}[\boldsymbol{\epsilon}|\mathbf{C}_A, \mathbf{C}_G - \mathbf{1}c_G] = 0$.

This is a reasonable model given "key 1," since "correlation" is inherently a measure of linear dependence. By the Strong Law of Large Numbers, $\lim_{m\to\infty}\frac{1}{m}\mathbf{1}\cdot\mathbf{C}_A = c_A$. Also, in finite samples of size m, $\mathrm{Var}[\frac{1}{m}\mathbf{1}\cdot\mathbf{C}_A] = \frac{1}{m}\mathrm{Var}[C_A]$. We claim, with control variates, that we can do better, in the sense of a smaller standard error of the estimate. The optimal coefficient (variance minimizer) of Equation (30) is the population parameter $\beta = \frac{\mathrm{Cov}[C_A,C_G]}{\mathrm{Var}[C_G]}$. Component-wise,

$$\mathrm{Var}[C_A^{\beta}] = \mathrm{E}[\big((C_A - E[C_A]) - \beta(C_G - c_G) + \epsilon\big)^2] = \mathrm{Var}[C_A] - 2\beta\mathrm{Cov}[C_A, \bar{C}_G] + \beta^2\mathrm{Var}[C_G] + \mathrm{Var}[\epsilon'],$$

where $\bar{C}_G \equiv C_G - c_G$. By the first-order condition, the claim follows and is verified by the second-order condition. On substitution of the minimizer: $\mathrm{Var}[C_A^{\beta}] - \mathrm{Var}[C_A] = -\frac{\mathrm{Cov}^2[C_A,C_G]}{\mathrm{Var}[C_G]} + \mathrm{Var}[\epsilon] = -\rho^2\mathrm{Var}[C_A] + \mathrm{Var}[\epsilon]$, where ρ is the correlation coefficient of C_A and C_G; and so, $\mathrm{Var}[C_A^{\beta}] = (1-\rho^2)\mathrm{Var}[C_A] + \mathrm{Var}[\epsilon]$. We conclude that if "key 1" holds *and* the model is a good representation of reality, i.e., $\mathrm{Var}[\epsilon]$ is small, then $\mathrm{Var}[C_A^{\beta}] < \mathrm{Var}[C_A]$.

In fact, we do not know the population parameters β or ρ; hence, we estimate them via standard regression methods. The *best linear unbiased estimator* of β is known [21] to be obtained from $(\hat{C}_A, \hat{\beta})^T = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{C}_A$, where $\mathbf{X} = (\mathbf{1}, \mathbf{C}_G)$. We then run a "warm-up" of the simulation and obtain via ordinary least squares regression the stand-in (see Equation (30)) for β (in terms of the stand-in $\hat{\rho}$) given by

$$\hat{\beta} = \frac{\mathrm{Cov}[\mathbf{C}_A, \mathbf{C}_G]}{\sqrt{\mathrm{Var}[\mathbf{C}_A]\mathrm{Var}[\mathbf{C}_G]}}\frac{\sqrt{\mathrm{Var}[\mathbf{C}_A]}}{\sqrt{\mathrm{Var}[\mathbf{C}_G]}} = \hat{\rho}\frac{\sqrt{\mathrm{Var}[\mathbf{C}_A]}}{\sqrt{\mathrm{Var}[\mathbf{C}_G]}}.$$

This procedure produces an implementable CV estimator, $C_A^{\hat{\beta}} - C_A = -\hat{\beta}(C_G - c_G) + e$, where e is an error term associated with the simulation. In our case, since the correlation between $C_A(T,k;\sigma)$ and $C_G(T,k;\sigma)$ is highly positive, we choose the a priori naive CV estimator $\hat{\beta} = 1$. $\square$

For the case of the BSM valuation, subject to the induced risk of estimator $\widehat{\sigma}_n$, we *cannot* directly use Equation (15). However, Algorithm 1 is applicable. For our valuation, the implementation of the CV approach requires the generation of comparable paths $(S(t;\sigma), t > 0)$ and $(S(t;\widehat{\sigma}_n), t > 0)$. One way to satisfy this requirement is via the use of common random numbers [23]. Having determined the df based on the estimation phase, and having decided on the number of paths, say ℓ, for the simulation, we also generate ℓ realizations of $\widehat{\sigma}_n^2$. We now use common random numbers, sequentially generating a pair of paths — one each of $(S(t;\sigma), t > 0)$ and $(S(t;\widehat{\sigma}_n), t > 0)$.

CHAPTER III

A LIKELIHOOD RATIO APPROACH TO HEDGING WITH ESTIMATION RISK

We examine contingent claim sensitivities associated with an assortment of European option portfolios that are based on an estimator for the volatility of the underlying geometric Brownian motion process. Our approach for evaluating the option sensitivities — the Greeks — uses the likelihood ratio method. This allows us to obtain computable results for the Greeks that depend on a new post-estimation representation of the underlying financial process. In the face of this inherent estimation risk, we are able to address many derivative portfolio pricing and hedging complications that potentially manifest themselves in the financial sector of a modern economy.

3.1 Introduction

The basic popular model for a stock price dynamic is the constant-coefficients geometric Brownian motion (GBM) process with volatility parameter σ. This is the specification utilized by Black and Scholes, [9] and Merton [41] (BSM) in their seminal work on options valuation. The ability to specify the correct value to σ is an essential element in understanding the time-series behavior of an assortment of financial markets, e.g., equities, foreign exchange rates, LIBOR rates, and the contingent claims associated with them. In this chapter we, as our precursors, Boyle and Ananthanarayanan [11], Butler and Schachter [15], and Ncube and Satchell [48], study the consequences of replacing the unknown volatility parameter by its natural estimator, when valuing European contingent claims. Specifically, unlike most work in this area that depends on the "law" of the unconscious statistician-quant (LUQ) [5], the procedure we propose for valuing and hedging financial instruments is more in line with how financial agents view options markets. This chapter briefly reviews our results and extends the analysis to the option sensitivities — the so-called Greeks.

We adhere to the framework discussed in Henrard [29] and Mykland [45, 46], where the concern is with the obvious (but usually ignored) fact that the true joint or marginal probability distributions governing the underlying securities are unknown. Additionally, we address two of the three sources of risk cited by Green and Figlewski [26] in their paper dealing with market and model risk. In particular, conditional on the correctness of the model, we explicitly consider their second source of risk which is linked to the unknown input parameters, e.g., σ in the BSM model, and their third risk source emanating from the chosen hedging technique, e.g., static versus any of a variety of time- and event-dependent techniques. Whereas, Henrard's [29] concern is with the "... study [of] the error coming from misestimation of the parameter of the model, not from the inadequacy of the model with reality," we suppose that the estimation technique is correct, but — subject to the usual draw of the data — unlikely to give the exact value for σ. Of equal importance, we further assume that economic agents are cognizant of this fact when making decisions. To this end, we use an explicitly adjusted version of basic GBM to model the underlying equity price, and in the process explore the effects of parameter estimation error on the modeled behavior of key financial variables — especially option valuations and the hedging practices that seek to avoid unfavorable chance outcomes in a portfolio. Our examples use an equity price though similar results apply to an assortment of other types of financially based stochastic processes.

The GBM model we propose — with estimated volatility — does not reflect the exact behavior of financial variables, but we are of the opinion that the proposed model is more in "tune" with how economic agents process information in many option markets. Specifically, the model is a reasonable depiction of reality when trader-speculator beliefs concerning volatility change are regarded as *unanticipated* — essentially both in timing and magnitude. That is market participants are subject to occasional intermittent and, in any sense, unpredictable shocks in volatility. Our tuned GBM model goes some way in indicating the appropriate methods for valuing and hedging future outcomes of claims when the effects of parameter uncertainty are internalized. In conjunction with our model, when deciding on a hedging strategy, we are able to ascertain the effects of parameter error distributions on the portfolio and its hedging structure. To accomplish this we apply the likelihood ratio method (Asmussen and Glynn [4]) (also called the score function method, Rubinstein and Shapiro [52]) to our post-estimation probability density function (p.d.f.) to obtain assorted option sensitivities.

Utilizing the likelihood ratio method, an extensive example is developed. It deals with a BSM vanilla call for which a closed-form solution exists. Correct decisions — in the sense of maintaining and enhancing portfolio value — in part depend on comprehending a map of parameter estimation error and where it can or cannot make a difference during a dynamic hedging process. We provide such a map. In summary, this paper's contribution is devoted to a better understanding of current models and modeling when there is less-than-complete, but nonetheless quantifiable knowledge about the true value of the volatility — as is the case when operating in a world where both quantifiable and pure types of uncertainty exist; see e.g., Knight [34], where quantifiable uncertainty is defined as risk, and pure uncertainty is the state where an economic agent lacks the ability or information necessary to assign probabilities to chance outcomes.

The paper is arranged in the following order. In §3.2 we briefly review the BSM valuation of a contingent claim based on an underlying GBM process. §3.3 deals with the consequences of incorporating the volatility estimator on the perceived valuation and hedging functions for a variety of European options. §3.4 introduces alternative general option valuation formulae along with associated option sensitivities (pre- and post-estimation), i.e., the Greeks. §3.5 considers dynamic hedging via option replication for both the pre- and post-estimation cases for portfolios consisting of vanilla claims. §3.6 concludes and offers suggestions for further research. The appendix in this chapter provides a further example dealing with dynamic hedging.

3.2 Basics

We now review some basic option formulae. All are based on the GBM constant-coefficients model of the price of an equity,

$$S(t;\sigma) \;\equiv\; s_0 \exp\left\{\left(\mu - \tfrac{\sigma^2}{2}\right)t + \sigma\mathcal{W}(t)\right\} \;\sim\; s_0 \exp\Big\{\mathrm{Nor}\Big(\left(\mu - \tfrac{\sigma^2}{2}\right)t, \sigma^2 t\Big)\Big\},\;\; t \geq 0, \tag{31}$$

where $s_0 \equiv S(0;\sigma)$ is the (known) initial price; $\mu \in \mathbb{R}$ and $\sigma > 0$ represent drift and volatility parameters specifying the empirical market measure driven by a standard Brownian motion (BM) process $(\mathcal{W}(t),\ t \geq 0)$ [23]. As Equation (31) stands, the only source of randomness emanating from $(S(t;\sigma), t \geq 0)$ is accounted for by the BM.

The classic BSM formulation of a contingent claim's value depends on the underlying equity unit value $S(t;\sigma)$ — viewed as a random variable realized at a given future time instant $t > 0$ whose outcome depends on the *known volatility* σ — all governed by the time-dependent law specified by the cumulative distribution function (c.d.f.) or probability density function (p.d.f.) given in the following well-known [30] result.

Lemma 3 The c.d.f. and p.d.f. of $S(t;\sigma)$ are

$$F_{S(t;\sigma)}(y) \ \equiv \ \Phi\left(\frac{\ell\mathrm{n}(\frac{y}{s_0}) - (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right), \quad y > 0 \qquad \text{and} \tag{32}$$

$$f_{S(t;\sigma)}(y) \ \equiv \ \frac{1}{y\sigma\sqrt{t}}\,\phi\left(\frac{\ell\mathrm{n}(\frac{y}{s_0}) - (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right), \quad y > 0, \tag{33}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the standard normal c.d.f. and p.d.f., respectively.

In the most-common cases, given any future time $t = T$, the random payoffs for a call and a put are respectively

$$C(s_0, k, T;\sigma) \ \equiv \ (S(T;\sigma) - k)^+ \ \text{ and } \ P(s_0, k, T;\sigma) \ \equiv \ (k - S(T;\sigma))^+,$$

where $x^+ = \max(x, 0)$ and k is the contingent contracts strike price. With $\tau \equiv T - t$ indicating the remaining time until expiry and $s_t \equiv S(t;\sigma)$ denoting the realized security value at time instant t, define the market observable $\boldsymbol{v}_t \equiv (s_t, k, \tau)$. Then the two contingent claim values $c(\boldsymbol{v}_t;\sigma) \ \equiv \ \mathrm{e}^{-r\tau}\mathrm{E}[C(\boldsymbol{v}_t;\sigma)]$ and $p(\boldsymbol{v}_t;\sigma) \ \equiv \ \mathrm{e}^{-r\tau}\mathrm{E}[P(\boldsymbol{v}_t;\sigma)]$ are related to one another, on arbitrage grounds, via the put-call parity relation [55],

$$p(\boldsymbol{v}_t;\sigma) \ = \ c(\boldsymbol{v}_t;\sigma) - s_t + k\mathrm{e}^{-r\tau}. \tag{34}$$

Each of $c(\boldsymbol{v}_t;\sigma)$ or $p(\boldsymbol{v}_t;\sigma)$ can individually be calculated with the aid of (32) or (33) by setting $\mu = r$, where r is the fixed risk-free interest rate. It is shown in [41] that all European claim types — independent of the law governing the underlying process — satisfy (34). For instance, by appropriately amending and reinterpreting the above, we can use an Asian call value (a call dependent on the average underlying price over some time period) to obtain the corresponding Asian put value.

At any moment in time, the contingent value $c(\boldsymbol{v}_t;\sigma)$ depends on the current aggregation of beliefs (objective and subjective), over the time interval $[t,T] \subset [0,T]$, of option market participants concerning the inherent variability of the tradable on which the contract is written. In particular, with remaining claim life τ, and with the specified values s_t, r, σ, and k, the call option value is

$$c(\boldsymbol{v}_t;\sigma) \;=\; s_t\,\Phi(z_{_+}) - k\,\mathrm{e}^{-r\tau}\,\Phi(z_{_-}), \tag{35}$$

where

$$z_{\pm} \;\equiv\; \frac{\ell\mathrm{n}\left(\frac{s_t}{k}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}. \tag{36}$$

The right-hand side of (35) is the classic BSM [55] formula for $c(\boldsymbol{v}_t;\sigma)$, interpreted as the present value of $\mathrm{E}[C(\boldsymbol{v}_t;\sigma)]$ at time t with expiry set at instant T; the expectation is taken with respect to the risk-neutral measure — the measure that requires all discounted (at the risk-free interest rate r) tradable claims to satisfy the martingale property [55].

3.3 Consequences of Estimating σ

This section reviews from Chapter 2 some of the differences encountered in valuation when we incorporate the frequentist estimator of volatility $\widehat{\sigma}_n$ in the BSM valuation model in place of the typically unknown volatility σ.

3.3.1 An Estimator for σ

We can estimate σ using data available during an "estimation period" occurring before the present time, say on the interval $[-n,\,0]$; and then at time $t=0$, use the estimate of σ to value a contingent claim with attached future expiry time T.

Under the assumption of a constant-coefficients GBM, it is reasonable to model the equity price during the estimation period $[-n,\,0]$ analogously to (31), i.e.,

$$\widetilde{S}(t;\sigma) \;\equiv\; \widetilde{S}(-n;\sigma)\,\exp\bigl\{\bigl(\mu - \tfrac{\sigma^2}{2}\bigr)(n+t) + \sigma\widetilde{\mathcal{W}}(n+t)\bigr\}, \quad -n \le t \le 0,$$

where $\widetilde{S}(-n;\sigma)$ is the equity price at time $-n$ and $(\widetilde{\mathcal{W}}(n+t),\ -n \le t \le 0)$ is a standard BM. As a factual matter, any increments from the estimation segment of the underlying BM are independent of the post-estimation segment $(\mathcal{W}(t),\ t \ge 0)$. So with no loss in generality, by dividing $[-n,\ 0]$ into n equal increments, we obtain the discretely sampled GBM process log-returns,

$$R_i \;\equiv\; \ell\mathrm{n}\left(\frac{\widetilde{S}(-n+i;\sigma)}{\widetilde{S}(-n+i-1;\sigma)}\right) \;=\; \mu - \tfrac{\sigma^2}{2} + \xi_i, \quad \text{for } i = 1, 2, \ldots, n, \tag{37}$$

where $\xi_i \equiv \sigma\big[\widetilde{\mathcal{W}}(i) - \widetilde{\mathcal{W}}(i-1)\big]$ for $i = 1, 2, \ldots, n$. By independent increments of BM, $R_1, R_2, \ldots, R_n$ are independent and identically distributed (i.i.d.) $\mathrm{Nor}(\mu - \frac{\sigma^2}{2}, \sigma^2)$ random variables. One can then use the sample variance [30] of the R_i's as a point estimator for σ^2, i.e.,

$$\widehat{\sigma}_n^2 \;\equiv\; \frac{1}{n-1}\sum_{i=1}^{n}(R_i - \bar{R}_n)^2 \;=\; \frac{1}{n-1}\sum_{i=1}^{n}(\xi_i - \bar{\xi}_n)^2 \;\sim\; \frac{\sigma^2\chi^2_{n-1}}{n-1}, \tag{38}$$

where $\bar{R}_n \equiv \sum_{i=1}^{n} R_i/n$ and $\bar{\xi}_n \equiv \sum_{i=1}^{n}\xi_i/n$ are the appropriate sample means, and χ^2_ε denotes a chi-square random variable with ε degrees of freedom (df). Thus, $\mathrm{E}[\widehat{\sigma}_n^2] = \sigma^2$, and we can regard $\widehat{\sigma}_n$ as a legitimate estimator of σ.

3.3.2 Results Concerning the Underlying Asset

Our results in Chapter 2 give the unconditional post-estimation c.d.f. and p.d.f. of the perceived equity process $(S(t;\widehat{\sigma}_n),\ t > 0)$ in terms of $\Phi(\cdot)$ and the χ^2_{n-1} p.d.f., $f_{\chi^2_{n-1}}(\cdot)$.

Lemma 4 The c.d.f. of $S(t;\widehat{\sigma}_n)$, $n \ge 2$, is

$$F_{S(t;\widehat{\sigma}_n)}(y) \;=\; \frac{n-1}{\sigma^2}\int_0^\infty \Phi\left(\frac{\ell\mathrm{n}(\frac{y}{s_0}) - \left(\mu - \frac{w}{2}\right)t}{\sqrt{wt}}\right) f_{\chi^2_{n-1}}\!\left(\tfrac{(n-1)w}{\sigma^2}\right)\mathrm{d}w, \;\; y > 0. \tag{39}$$

Lemma 5 The p.d.f. of $S(t;\widehat{\sigma}_n)$, $n \ge 2$, is

$$f_{S(t;\widehat{\sigma}_n)}(y) \;=\; \frac{K}{y}\exp\left\{-\frac{a_0(y)}{2}\right\}\int_0^\infty \exp\left\{-\left(\frac{a_1(y)}{w} + a_2 w\right)\right\} w^{\frac{n-4}{2}}\,\mathrm{d}w, \tag{40}$$

where we define the functions $a_0(y) \equiv \ell\mathrm{n}(\frac{y}{s}) - \mu t$ and $a_1(y) \equiv \frac{a_0^2(y)}{2t}$, and the positive numbers $a_2 \equiv \frac{t}{8} + \frac{n-1}{2\sigma^2}$ and $K \equiv \left(\frac{n-1}{2\sigma^2}\right)^{\frac{n-1}{2}}\left(\Gamma(\frac{n-1}{2})\sqrt{2\pi t}\right)^{-1}$ with $\Gamma(\cdot)$ the gamma function.

We showed in Chapter 2, that for any time $t \ge 0$, the mean of the post-estimation random variable $S(t;\widehat{\sigma}_n)$ in Lemma 4 is the same as the mean of the underlying financial $S(t;\sigma)$, i.e., $\mathrm{E}[S(t;\sigma)] = \mathrm{E}[S(t;\widehat{\sigma}_n)] = s_0\mathrm{e}^{\mu t}$, where the second expectation is with respect to a joint distribution.

3.4 European Claims Under the Post-Estimation Law

This section provides the necessary fundamental formulae that are needed in §3.5 when we discuss strategies for hedging European option portfolios — subject to the estimated volatility — by "manufacturing" a hedge that is a linear combination of a money market instrument and the underlying equity. We start by briefly reviewing our approach to pricing contingent claims. Next, we derive and discuss the associated option sensitivities, which are used to implement any of a large class of hedging strategies.

3.4.1 Vanilla Calls

An alternative to the standard pricing formula, especially useful for computing valuations governed by the post-estimation laws, is considered next. The idea is illustrated via valuation of a vanilla call. The c.d.f. of the post-estimation vanilla European call option $C(\boldsymbol{v}_0; \widehat{\sigma}_n)$ is given by

$$F_{C(\boldsymbol{v}_0;\widehat{\sigma}_n)}(y) \;=\; \begin{cases} 0 & \text{if } y < 0 \\ F_{S(T;\widehat{\sigma}_n)}(y+k) & \text{if } y \geq 0. \end{cases} \tag{41}$$

The call has a point probability at $y = 0$ equal to $F_{S(T;\widehat{\sigma}_n)}(k)$ — the probability of being out-of-the-money (OTM) at the time of expiry. The present value of the call $C(\boldsymbol{v}_0; \widehat{\sigma}_n)$ is

$$c(\boldsymbol{v}_0; \widehat{\sigma}_n) \;=\; \mathrm{e}^{-rT}\mathrm{E}[C(\boldsymbol{v}_0; \widehat{\sigma}_n)] \;=\; \mathrm{e}^{-rT} \int_k^{\infty} \bar{F}_{S(T;\widehat{\sigma}_n)}(y)\,\mathrm{d}y, \tag{42}$$

with $\bar{F}_{S(T;\widehat{\sigma}_n)}(y) \equiv 1 - F_{S(T;\widehat{\sigma}_n)}(y)$.

3.4.2 The Likelihood Ratio and the Greeks

This section presents option sensitivities ("Greeks"), used in quantifying and comparing hedging strategies. Of the many possible sensitivities [23] associated with individual European claims, the most important and informative ones are the first-order Greeks delta ($\delta(\boldsymbol{v}_t; \star) \equiv \frac{\partial c(\boldsymbol{v}_t;\star)}{\partial s_t}$) and vega ($\vartheta(\boldsymbol{v}_t; \star) \equiv \frac{\partial c(\boldsymbol{v}_t;\star)}{\partial \sigma}$), for $\star = \sigma$ or $\widehat{\sigma}_n$. Table 6 lists the formulae for these BSM option Greeks with known volatility on an underlying equity that pays no dividends [23].

1. Delta is the most-frequently used Greek for monitoring and implementing a hedge. It gives the sensitivity (partial derivative) of the option value with respect to an

Table 6: Classic BSM Greeks

Greek	call formula	put formula
δ	$\Phi(z_+)$	$\Phi(z_+) - 1$
ϑ	$s_t\phi(z_+)\sqrt{\tau}$	$s_t\phi(z_+)\sqrt{\tau}$

incremental and unknown change in the currently observed price of the underlying on which the contingent claim is written.

2. Vega gives the sensitivity of the option value with respect to an exogenous, i.e., unexpected, change in σ — a very useful sensitivity whether or not σ is known.

Extension of our analysis to other useful first (rho, theta) and second (gamma and cross-partials) order sensitivities is obvious, and so will not be pursued in the remainder of the paper.

Let us review some results needed to calculate Greeks via the *likelihood ratio* [23] or *score function* method [52]. In our case, the likelihood ratio approach is applied to obtain the Greeks because for certain claim types there are no known BSM closed-form expressions for valuation formula that depend on the post-estimation c.d.f. of Lemma 4. The alternative of direct differentiation can be problematic since evaluating the Greeks requires the numerical calculation of double and triple integrals with, in this case, assorted numerical stability problems. In this sense, the likelihood ratio method is more general and can be applied to many contingent payoff types, e.g., barrier and Asian claims [23].

At a given point in time the Greeks, in the pre-estimation case, are defined by a vector dependent on the p.d.f. of the underlying equity, which we now write as $f_{S(\tau;\sigma)}(y; s_t, \sigma)$ to emphasize the parameters s_t and σ. The p.d.f., by definition, is linked to the score function via the mapping $\boldsymbol{\mathcal{S}}^\sigma : \mathbb{R}^3 \to \mathbb{R}^2$, $\boldsymbol{\mathcal{S}}^\sigma(y; s_t, \sigma) \equiv \nabla \ell\mathrm{n}\big[f_{S(\tau;\sigma)}(y; s_t, \sigma)\big]$, where ∇ is the gradient operator acting on parameters s_t and σ. Option sensitivities can be calculated for any payoff function $\varphi(y; k)$ that is independent of the population parameters instantiating the p.d.f., e.g., $\varphi(y; k) = \mathrm{e}^{-r\tau}(y - k)^+$, which does not depend on s_t and σ. Given the payoff and score functions, the BSM Greeks are

$$(\delta, \nu) \quad = \quad \mathrm{E}[\boldsymbol{\mathcal{S}}^\sigma(Y; s_t, \sigma)\varphi(Y; k)] \; = \; \int_0^\infty \boldsymbol{\mathcal{S}}^\sigma(y; s_t, \sigma)\varphi(y; k) f_{S(\tau;\sigma)}(y; s_t, \sigma)\, \mathrm{d}y.$$

As a basis for comparison between the pre- and post-estimation densities of the underlying, we provide in Proposition 2 the score function of the classic (Lemma 3) BSM p.d.f. The European option sensitivities in this classic case are obtained by straightforward differentiation.

Proposition 2 Assuming σ to be known and using the p.d.f. in Equation (33), the BSM first-order score function components are $\boldsymbol{\mathcal{S}}^{\sigma}(y;s_t,\sigma) \equiv \big(\mathcal{S}_s^{\sigma}(y;s_t,\sigma), \mathcal{S}_\sigma^{\sigma}(y;s_t,\sigma)\big)$, where

$$\begin{aligned}
\mathcal{S}_s^{\sigma}(y;s_t,\sigma) &\equiv \frac{\partial \ell\mathrm{n}\big[f_{S(\tau;\sigma)}(y;s_t,\sigma)\big]}{\partial s_t} = \frac{\ell\mathrm{n}\big(\frac{y}{s_t}\big) - \big(\mu - \frac{\sigma^2}{2}\big)\tau}{s_t\tau\sigma^2}, \\
\mathcal{S}_\sigma^{\sigma}(y;s_t,\sigma) &\equiv \frac{\partial \ell\mathrm{n}\big[f_{S(\tau;\sigma)}(y;s_t,\sigma)\big]}{\partial \sigma} = \frac{4\ell\mathrm{n}\big(\frac{y}{s_t}\big)(\ell\mathrm{n}\big(\frac{y}{s_t}\big) - 2\tau\mu) - 4\tau\sigma^2 - \tau^2\left(\sigma^4 - 4\mu^2\right)}{4\tau\sigma^3}.
\end{aligned}$$

In the post-estimation case, we may also include the sensitivity with respect to n, which can be regarded as a parameter (not necessarily integer) related to the amount of "available information" utilized to implement a contingent claim valuation. This view of n is left to a subsequent paper dealing, under various information structures, with what is the "optimal" n in a market or general equilibrium context. The first-order sensitivities under the post-estimation p.d.f. of Lemma 5 are given by the mapping $\boldsymbol{\mathcal{S}}^{\widehat{\sigma}_n} : \mathbb{R}^4 \to \mathbb{R}^2$ via

$$\boldsymbol{\mathcal{S}}^{\widehat{\sigma}_n}(y;s_t,\sigma,n) \equiv \big(\mathcal{S}_s^{\widehat{\sigma}_n}(y;s_t,\sigma,n), \mathcal{S}_\sigma^{\widehat{\sigma}_n}(y;s_t,\sigma,n)\big).$$

It turns out that these sensitivities can be written in representations employing repeated use of the modified Bessel function of the second kind $\mathcal{K}_\alpha[\beta]$. In order to simplify the exposition, we use the following notation in the specification of the post-estimation score function.

- $B_0(y) \equiv r\tau - \ell\mathrm{n}\big(\frac{y}{s_t}\big)$, $B_{01} \equiv \sqrt{\frac{4(n-1)+\sigma^2\tau}{\tau}}$, $B_1(y) \equiv \frac{1}{2\sigma}B_{01}|B_0(y)|$, and the sign function $\mathrm{sgn}(x) = 1$ if $x > 0$, -1 if $x < 0$, and 0 otherwise.

With the aid of Lemma 5, we obtain the following result.

Proposition 3 When σ is unknown and the estimate $\widehat{\sigma}_n$ is used as a stand-in, the BSM first-order score function components with respect to s_t and σ are

$$\begin{aligned}
\mathcal{S}_s^{\widehat{\sigma}_n}(y;s_t,\sigma,n) &\equiv \frac{\partial \ell\mathrm{n}\big[f_{S(\tau;\widehat{\sigma}_n)}(y;s_t,\sigma,n)\big]}{\partial s_t} \\
&= \frac{1}{2s_t}\left(1+\frac{2(n-2)}{B_0(y)}-\frac{B_{01}\mathcal{K}_{\frac{n}{2}}\big[B_1(y)\big]\mathrm{sgn}(B_0(y))}{\sigma\mathcal{K}_{\frac{2-n}{2}}\big[B_1(y)\big]}\right), \\
\mathcal{S}_\sigma^{\widehat{\sigma}_n}(y;s_t,\sigma,n) &\equiv \frac{\partial \ell\mathrm{n}\big[f_{S(\tau;\widehat{\sigma}_n)}(y;s_t,\sigma,n)\big]}{\partial \sigma} = (n-1)\left(\frac{2|B_0(y)|\mathcal{K}_{\frac{n}{2}}\big[B_1(y)|\big]}{\tau\sigma^2B_{01}\mathcal{K}_{\frac{2-n}{2}}\big[B_1(y)\big]}-\frac{1}{\sigma}\right).
\end{aligned}$$

3.5 Pre- and Post-Estimation Replication Regimes

With the above Greeks, we proceed to discuss the ramifications for hedging in the pre- and post-estimation cases. In §3.5.1, our initial discussion is of a general nature reviewing the classic replication argument. This is followed in §3.5.2 by specific examples of monitoring and hedging option portfolios where a point estimate to volatility is assigned, and in addition, where the risk of that point estimate is incorporated in the pricing–hedging policy.

3.5.1 Delta Hedging

It is well known that a European option in the BSM model has a value that can be replicated, i.e., synthetically constructed, by *continuously* trading in the underlying equity and a money market account; and that this idea is used to construct an assortment of dynamic — i.e., time- and event-dependent — hedging schemes. Such constructed schemes are often referred to as *delta hedging* [6] and at each moment in time result in the accumulated profits and losses (P&L), under continuous hedging, being equal to the option value (cost), determined via the calculation of the risk-neutral expectation.

There exist two broad classes of hedging schemes. Essentially, the *dynamic* hedge is concerned with a mark-to-market profile of profits and losses. On the other hand, one can also construct a *static* hedge, whose concern is with the balance of value obtained at the time of expiry; in the interim for such a hedge, it is possible that P&L can exceed any given wealth buffer. In terms of static hedging, Green and Figlewski [26] note “the strategy of writing and holding option positions without hedging entails [a] very large risk of exposure.” Similar to Carr [16], who asks whether one should hedge at historical or implied volatility, we are concerned with what happens when estimation risk is incorporated into the valuation and hedging procedure.

In the next subsections, we review general delta hedging principles, and then give a detailed discrete example illustrating a dynamic hedging procedure that incorporates the randomness emanating from the estimation component of volatility.

3.5.1.1 Hedging Strategy in Complete Markets

In the following discussion, where we review the hedging strategy for a BSM call option, the required technical conditions on the integrability of the various involved functions hold, e.g., see [55]. We work in the context of a *complete* market, i.e., roughly speaking, the number of possible sources of randomness reflected in the financial market equals the number of underlying securities on which (non-trivial) contingent claims can be written. At time instant $t \in [0, T]$, denote by $\delta(t)$ the number of units of the underlying equity, with unit value $S(t;\sigma)$; and let $\varrho(t)$ be the number of units of the numeraire money market held, each valued at $D(t) \equiv \mathrm{e}^{rt}$. Thus, the portfolio at time t has value $V(t) = \delta(t)S(t;\sigma) + \varrho(t)D(t)$. We require that the portfolio weights $\big((\delta(t), \varrho(t)),\, t \geq 0\big)$ be adapted processes of time, i.e., processes measurable with respect to a filtration satisfying $\mathcal{F}_t \supset \mathcal{F}_{t'} \supset \mathcal{F}_0$, $\forall t > t'$, where $\mathcal{F}_t$ is the induced sigma-algebra generated by the BM $(\mathcal{W}(s),\, 0 \leq s \leq t)$ [35]. The filtration is common to all agents and tells us how "market" information disseminates over time.

The portfolio at moment t is called *self-financing* if the need for an external source of funds to cover or withdraw from the value of the contingent claim has zero probability. That is, the above portfolio can change value only on account of capital appreciation; and therefore, one cannot increase (decrease) simultaneously both components of the portfolio, though one can change either at the expense of the other. This requirement is met by applying Itô's rule [35] to the portfolio $V(t)$, and in the process discovering that $\big(S(t;\sigma) + \mathrm{d}S(t;\sigma)\big)\mathrm{d}\delta(t) + \big(D(t) + \mathrm{d}D(t)\big)\mathrm{d}\varrho(t) = 0$, for all $t \in [0, T]$, with probability one, is a sufficient condition for self-financing [55]. Summing up, for a self-financed portfolio,

$$\mathrm{d}V(t) \;=\; \delta(t)\mathrm{d}S(t;\sigma) + \varrho(t)rD(t)\mathrm{d}t. \tag{43}$$

Now with $\mu = r$, set $S^*(t;\sigma) \equiv S(t;\sigma)/D(t)$ and $V^*(t) \equiv V(t)/D(t)$. On arbitrage grounds, the folk tale is that relative prices of financials must be martingales, and so the deflated processes $(S^*(t;\sigma),\, t \geq 0)$ and $(V^*(t),\, t \geq 0)$ are martingales. This is the process version we work with below to construct the appropriate replicating portfolio weights.

Amend Equation (43) to its martingale version and continue to maintain the property of self-financing. Then in integrated form, where the last term follows by Itô's Lemma [55] and risk-neutrality, we have

$$V^*(t) \;=\; V(0) + \int_0^t \delta(u)\mathrm{d}S^*(u;\sigma) \;=\; V(0) + \int_0^t \delta(u)S^*(u;\sigma)\sigma\mathrm{d}\mathcal{W}(u). \tag{44}$$

So the only changes in the normalized portfolio value $V^*(t)$ come about through an exogenously given source of capital change $\mathrm{d}S^*(u;\sigma)$. Since $V^*(t)$ is a martingale, it satisfies the Martingale Representation Theorem (see [55]), which states that for any BM-driven martingale $(M(t),\, t \geq 0)$, there exists an adapted process $(\theta(u),\, u \geq 0)$ such that the integral representation $M(t) = M(0) + \int_0^t \theta(u)\mathrm{d}\mathcal{W}(u)$ holds. The expectation of the portfolio value in Equation (44), under the risk-neutral measure, is therefore $\mathrm{E}[V^*(t)|\, \mathcal{F}_0] = V(0)$. Also, by the above, the connection between the fair price of a call and the synthetically constructed portfolio that replicates it is given by

$$\frac{C(\boldsymbol{v}_t;\sigma)}{D(\tau)} \;=\; M(0) + \int_0^\tau \theta(u)\mathrm{d}\mathcal{W}(u),$$

for some $(\theta(u),\, u \geq 0)$. To obtain the hedging process, use Equation (44) and the construction $\theta(u) = \delta(u)S^*(u;\sigma)\sigma$, $0 \leq u \leq t$. Hence, over $t \in [0,T]$, we have obtained a replication procedure, i.e., an adapted process of portfolio weights $\big((\delta(t), \varrho(t)),\, t \geq 0\big)$. We choose from Table 6 the portfolio weight process for the underlying to be $\delta(t) \equiv \Phi(z_+)$, $0 \leq t \leq T$. With this process, it can be shown that the contingent claim is attained [35], i.e., the option process is mimicked.

3.5.1.2 Example

We illustrate the traditional hedging/replication process of BSM and compare the result, via a detailed example in Table 7, to the case of post-estimation hedging. The two post-estimation deltas are those obtained by using our likelihood function characterization of delta and a naive plug-in delta. Naive hedging merely plugs the estimate $\widehat{\sigma}_n$ into a particular

BSM Greek. In Chapter 2 we derived the c.d.f. of such an estimator for δ, i.e., $\widehat{\delta} \equiv \Phi(\widehat{z}_+)$, where from Equation (36), $\widehat{z}_+$ is z_+ with $\widehat{\sigma}_n$ in place of σ. In the case of a vanilla call and its accompanying delta, using the LUQ mimics our post-estimation results. In the previously chapter we showed that the LUQ call, with expiry date τ years hence, is obtained by amending the BSM formula to $c^*(\boldsymbol{v}_t; \widehat{\sigma}_n) \equiv \frac{n-1}{\sigma^2}\mathrm{E}\big[s_t\,\Phi\big(z_+(\Xi)\big) - k\,\mathrm{e}^{-r\tau}\,\Phi\big(z_-(\Xi)\big)\big]$, where $z_\pm(\Xi) \equiv \frac{\ell\mathrm{n}(\frac{s_t}{k})+(r\pm\Xi/2)\tau}{\sqrt{\Xi\tau}}$ and $\Xi \sim \frac{\sigma^2}{n-1}\chi^2_{n-1}$. An analogous procedure gives us the LUQ delta $\delta^*(\boldsymbol{v}_t; \widehat{\sigma}_n)$. This call and its delta match our results exactly. However, extending the use of the LUQ to other Greeks — for instance, vega — breaks the connection to the post-estimation case.

For this example, assume that the market measure has known drift $\mu = 0.1$, the risk-free interest rate is $r = 0.05$, volatility is $\sigma = 1$, and $s_0 = k = 10$. Set the time increments so that they satisfy $\Delta = t_i - t_{i-1} = 1/52$, $i = 1, 2, \ldots, m$, where we consider an $m = 8$ week hedging experiment based on $T = 1/6$ (two months). It follows that $S(t_i; 1) = S(t_{i-1}; 1)\,\mathrm{e}^{(0.1-0.5)\Delta+\sqrt{\Delta}\,Z_i}$, with $Z_1, Z_2, \ldots, Z_m$ a sequence of i.i.d. Nor(0,1) random variables generating the underlying prices in row 3. Row 4, in line with Equation (37), reproduces the log-returns $(R(t_i;\sigma) \equiv R_i)$ for the particular drawing of GBM from the above instantiation. By Equation (38), for our sample draw of 10 observations, 1.019 is the annualized volatility over the entire (pre- and post-estimation) segment. Row 5 is a "running" tally of annualized volatility inclusive of all previous Δ-periods and starting in the pre-estimation phase consisting of a total of 3 periods, with volatility realization $\widehat{\sigma}_0 = 1.094$.

To aid clarity, in terms of what a priori state-of-the-world is dictating the hedging policy, we explicitly indicate $\delta(\boldsymbol{v}_{t_i}; \star)$, $\star = \sigma$ for the BSM delta, and $\delta(\boldsymbol{v}_{t_i}; \star)$, $\star = \widehat{\sigma}_n$ when the post-estimation delta is referenced — with likewise notation applied to the remaining hedging functions. Hedges take place, with zero transaction costs, on a discrete mesh composed of m time increments of size Δ. Since for $i = 1, 2, \ldots, m$, $(S(0;\sigma), S(\Delta;\sigma), S(2\Delta;\sigma), \ldots, S(i\Delta;\sigma))$ generates $\mathcal{F}_{i\Delta} \subset \mathcal{F}_{(i+1)\Delta}$, we know all necessary information, including $\mathrm{d}S(i\Delta;\sigma) \equiv S(i\Delta;\sigma) - S((i-1)\Delta;\sigma)$, $i = 1, 2, \ldots, m$, at each specified time point to implement a re-hedge.

In rows 6–12, the hedging policy illustrated assumes that agents know and use the true σ — this is the traditional discrete BSM hedge. Assume that a vanilla call is sold at a price

Table 7: Discrete vs. Continuous Hedging of a (Short) Call

1	end of week i	-2	-1	0	1	2	3	4	5	6	7	8
2	$\tau_i = \frac{9}{52} - t_i$			8	7	6	5	4	3	2	1	0
3	$S(t_i;\sigma)$	10.202	11.560	10	9.557	10.833	12.595	10.866	9.524	8.076	9.434	9.390
4	$R(t_i;\sigma)$	0.110	0.125	(0.145)	(0.045)	0.125	0.151	(0.148)	(0.132)	(0.165)	0.155	(0.005)
5	$\widehat{\sigma}_n$ ($\sigma = 1$)			1.094	0.933	0.888	0.865	0.962	0.976	1.000	1.019	
6	$c(\boldsymbol{v}_{t_i};\sigma)$			1.653	1.306	1.966	3.160	1.747	0.824	0.192	0.452	0
7	$\delta(\boldsymbol{v}_{t_i};\sigma)$			0.589	0.537	0.663	0.811	0.671	0.485	0.206	0.410	1
8	$\delta(\boldsymbol{v}_{t_i};\sigma)S(t_i;\sigma)$			5.888	5.134	7.182	10.211	7.288	4.619	1.666	3.870	
9	$\delta(\boldsymbol{v}_{t_i-\Delta};\sigma)S(t_i;\sigma)$				5.628	5.819	8.350	8.809	6.388	3.917	1.947	3.852
10	$\varrho(\boldsymbol{v}_{t_i};\sigma)$			(4.236)	(3.828)	(5.215)	(7.051)	(5.541)	(3.795)	(1.475)	(3.418)	
11	$\varrho(\boldsymbol{v}_{t_i-\Delta};\sigma)\mathrm{e}^{r\Delta}$				(4240)	(3.832)	(5.220)	(7.058)	(5.546)	(3.780)	(1.476)	(3.421)
12	P&L(σ)				0.082	0.021	(0.030)	0.004	0.018	(0.074)	0.019	0.431
13	$c(\boldsymbol{v}_{t_i};\widehat{\sigma}_n)$			1.659	1.126	1.753	3.006	1.666	0.770	0.190	0.449	0
14	$\delta(\boldsymbol{v}_{t_i};\widehat{\sigma}_n)$			0.591	0.514	0.682	0.852	0.684	0.473	0.192	0.403	1
15	$\delta(\boldsymbol{v}_{t_i};\widehat{\sigma}_n)S(t_i;\sigma)$			5.913	4.917	7.383	10.731	7.434	4.506	1.549	3.799	
16	$\delta(\boldsymbol{v}_{t_i-\Delta};\widehat{\sigma}_n)S(t_i;\sigma)$				5.651	5.573	8.584	9.257	6.516	3.821	1.809	3.782
17	$\varrho(\boldsymbol{v}_{t_i};\widehat{\sigma}_n)$			(4.254)	(3.790)	(5.631)	(7.725)	(5.767)	(3.736)	(1.358)	(3.350)	
18	$\varrho(\boldsymbol{v}_{t_i-\Delta};\widehat{\sigma}_n)\mathrm{e}^{r\Delta}$				(4.258)	(3.794)	(5.636)	(7.733)	(5.773)	(3.740)	(1.359)	(3.353)
19	P&L($\widehat{\sigma}_n$)				0.267	0.026	(0.057)	(0.142)	(0.027)	(0.109)	0.000	0.428
20	$\widehat{c}(t_i)$			1.965	1.118	1.667	2.877	1.659	0.776	0.192	0.477	0
21	$\widehat{\delta}(t_i)$			0.604	0.527	0.649	0.799	0.667	0.482	0.206	0.412	1
22	$\widehat{\delta}(t_i)S(t_i;\sigma)$			6.044	5.040	7.032	10.068	7.244	4.595	1.667	3.882	
23	$\widehat{\delta}(t_i-\Delta)S(t_i;\sigma)$				5.777	5.713	8.175	8.685	6.349	3.896	1.947	3.864
24	$\widehat{\varrho}(t_i)$			(4.080)	(3.922)	(5.364)	(7.191)	(5.585)	(3.819)	(1.475)	(3.405)	
25	$\widehat{\varrho}(t_i-\Delta)\mathrm{e}^{r\Delta}$				(4.084)	(3.926)	(5.370)	(7.198)	(5.590)	(3.823)	(1.476)	(3.409)
26	$\widehat{\text{P\&L}}$				0.575	0.121	(0.071)	(0.171)	(0.017)	(0.119)	(0.006)	0.456

(a) Pre-estimation phase: weeks $i = -2, -1, 0$.

(b) Parenthesis around any value indicate a negative number, i.e., accounting convention.

(c) Volatility is an annualized value, in this weekly case, by multiplying by the factor $\sqrt{52}$.

of $c(10, 10, 1/6; 1) = 1.653$, i.e., with an expiry date 8 weeks off the shorted call is valued so that the implied volatility equals the true volatility $\sigma = 1$. To start with, we initialize with a short call position $c(\boldsymbol{v}_0; \sigma)$ balanced by an equal position in value of $\delta(\boldsymbol{v}_0; \sigma)s_0 + \varrho(\boldsymbol{v}_0; \sigma)$, i.e., we are long in underlying value equal to $\delta(\boldsymbol{v}_0; \sigma)s_0$ plus short in the money market equal to $\varrho(\boldsymbol{v}_0; \sigma) = c(\boldsymbol{v}_0; \sigma) - \delta(\boldsymbol{v}_0; \sigma)s_0$. An advantage of shorting and then delta hedging a call in the market for the underlying is that one is likely to get an "easy fill" on orders since our equity component order flow will be "counter to the market" for the underlying, i.e., buy the equity in a falling market and sell it in a rising market. Rows 6–8 and 10 provide the continuously replicated portfolio values for the call option. Row 9 reflects the carried over value of the delta hedge $\delta(\boldsymbol{v}_{t-\Delta}; \sigma)S(t; \sigma)$, and row 11 is the debt (with interest) owed from the previous period of trading and hedging. Row 12 reflects the periodic profits from the weekly hedging, where

$$\text{P\&L}(\sigma) \equiv -c(\boldsymbol{v}_t; \sigma) + \delta(\boldsymbol{v}_{t-\Delta}; \sigma)S(t; \sigma) + \varrho(\boldsymbol{v}_{t-\Delta}; \sigma)\mathrm{e}^{r\Delta}, \tag{45}$$

i.e., P&L $\equiv$ (short call) + (stock hedge) + debt, where debt ≤ 0.

Table 7 incorporates a pre-estimation phase where we have bootstrapped [21] (with small perturbation error) three log-returns from the post-estimation period. This procedure provides some sense of consistency, at the expense of ever-so-slight correlation, between the two phases. In our example, the log-returns 0.110, 0.125, and (0.145) consist of the draw. In the "pre-estimation" phase we initialize volatility on $n = 3$ return observations. Thereafter, we update according to the formulae $\bar{R}_{i+1} = \frac{i}{i+1}\bar{R}_i + \frac{1}{i+1}R_{i+1}$ and $\widehat{\sigma}^2_{i+1} = \frac{i-1}{i}\widehat{\sigma}^2_i + \frac{1}{i}(R_{i+1} - \bar{R}_{i+1})^2, \quad i \geq 2$. Of course, there is no updating in the "known" volatility case since *all* relevant information needed to execute the replicating hedge is available. However, when the volatility is unknown, we update $c(\boldsymbol{v}_t; \widehat{\sigma}_n)$ and $\delta(\boldsymbol{v}_t; \widehat{\sigma}_n)$, at each instance prior to the activation of a hedging event, to $c(\boldsymbol{v}_{t+\Delta}; \widehat{\sigma}_{n+1})$ and $\delta(\boldsymbol{v}_{t+\Delta}; \widehat{\sigma}_{n+1})$.

We now complete, for the purpose of comparison to the traditional known-σ case, the hedge-replicate of the short call assuming that volatility is unknown, but where estimates may be calculated, and where we pursue a policy of re-balancing weekly. Rows 13–19 depict the hedging policy using the estimates of σ obtained from $\widehat{\sigma}_n$ in row 5. The calculations for each entry in rows 13–19 are the same as in the case of known σ. Here, when engaged in discrete hedging, we know that the replicating portfolio is not self-financing. This is confirmed by the P&L time series in row 19.

The final comparison of hedging policies uses the naive plug-in of $\widehat{\sigma}_n$ and is displayed in rows 20–26. Once again the updated $\widehat{\sigma}_n$ is used, as a plug-in in this case, to obtain a hedging sample time series.

Some tentative highlights, based on one path realization, among the three hedging policies are:

1. As one would anticipate, in terms of the smallest amplitude of P&L, BSM dominates the post-estimation case, which in turn dominates the case of the plug-in estimator.

2. The most significant differences in P&L tend to occur in the early hedging phase. This is, as well reasonable, since in terms of $n \geq 2$, $\widehat{\sigma}_n$ is a progressively better estimator of σ.

3. The plug-in policy tends to produce losses of approximately twice the P&L of the post-estimation policy, which in turn is about twice that of the BSM policy.

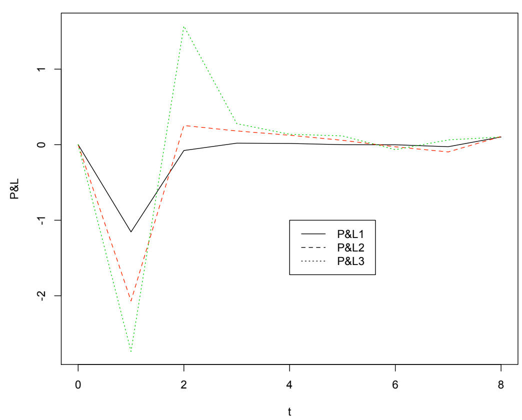
4. In terms of valuations over time, all policies appear to converge to the BSM policy. This is in accord with our theoretical results.

In an upcoming paper we construct an extensive simulation comparing the three states-of-the-world among alternative vanilla and exotic contingent claim types.

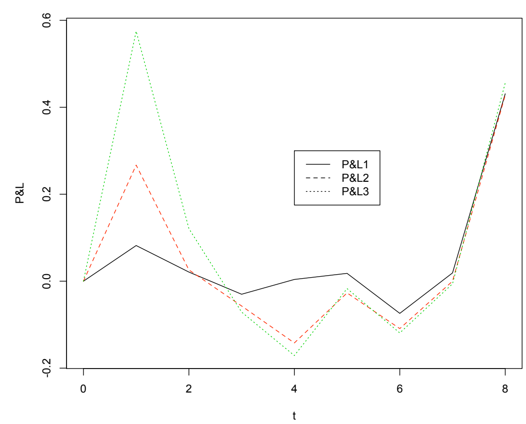
In the following discussion Figure 9 shows the hedging error associated with the three types of discrete replication based on the policies depicted in Table 7. Panel (b) graphs the P&L's referenced in Table 7, where we recall that the realized volatility is 1.019 and the option finishes ITM. In panel (a), the P&L time series have realized (sample) volatility of 1.476, and for panel (c), the P&L's realized (sample) volatility is 0.685. For both of these panels the option finishes OTM. In all of the three cases, the true volatility is $\sigma = 1$. The simulations depicted in panels (a)–(c) provide evidence confirming the amplitude relationship among the possible hedging policies. Table 8 gives the annualized P&L volatilities for the respective panels of Figure 9, subject to the just-described conditions.

Table 8: P&L Annualized Sample Volatility

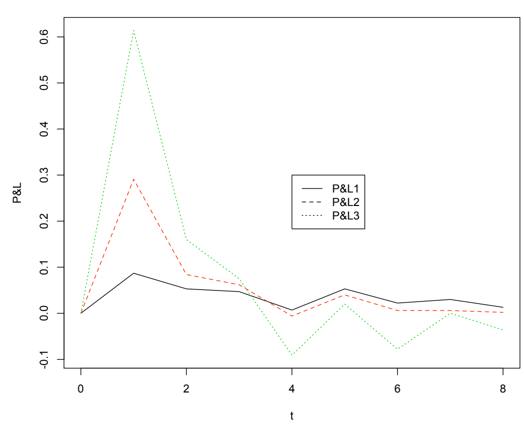
	P&L 1: $\widehat{\sigma}_n \gg \sigma$	P&L 2: $\widehat{\sigma}_n \approx \sigma$	P&L 3: $\widehat{\sigma}_n \ll \sigma$
P&L(σ)	2.971	1.131	0.189
P&L($\widehat{\sigma}_n$)	5.553	1.423	0.710
$\widehat{\text{P\&L}}$	8.640	1.981	1.656



(a) P&L High Realized Volatility



(b) P&L Same Realized Volatility



(c) P&L Low Realized Volatility

Figure 9: P&L: (a) Realized Vol. $\geq \sigma$; (b) Realized Vol. $\approx \sigma$; (c) Realized Vol. $\leq \sigma$.

3.5.2 Delta Hedged Portfolios

With the previous discussion in mind, this section illustrates by way of several representative examples of vanilla portfolios, that there exist substantial differences in delta hedging when comparing pre- and post-estimation delta. Recall that delta indicates the number of units of the underlying that one sells or purchases so as to maintain a constant portfolio value. Henrard's [29] comment concerning estimation risk is

> " [N]ot only one sells the option at the wrong price but also that the hedging used will be δ-neutral with respect to an incorrect δ. ... Even if all the option trades are done internally within a bank and each component of the bank believes that its risk is non-existent, the total profit of the bank can be [negative]."

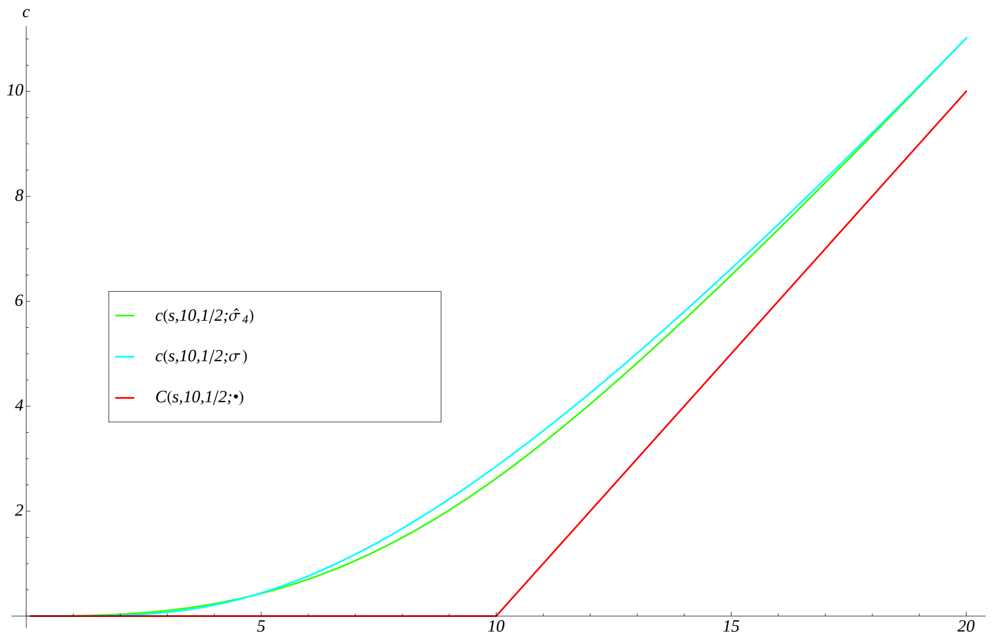
The pre- and post- differences extend throughout the Greeks and include the sensitivity vega, which is a component that warrants attention in any serious hedging strategy where volatility can change unexpectedly.

The following discussion takes as pairs Figures 10–11, Figures 12–13, Figures 14–15, and Figures 16–17. The diagrams present progressively more-complex hedging portfolios. All figure ensembles consider potential hedging-valuation time profiles with an initial duration of six months. The initial conditions for the profiles are exhibited in panel (a) of the even numbered figures. For the initial pairs, Figures 10 − 11 and Figures 12 − 13, the chosen instantiating parameters are $\sigma = 1$, $s = 10$, $k = 10$, $n = 4$, $r = 0.05$, and $T = 1/2$. Turning to the odd numbered figures, each produces the pre- and post-estimation delta contours along with three representative GBM price paths. The strategy over time in all the examples is that we sell the initial portfolio, e.g., a call, and then dynamically hedge along the way to expiry by buying (selling) $\delta(\boldsymbol{v}_{t_i}; \star)$, $i = 1, 2, \ldots, N$, units of the underlying stock, where N is the number of hedging opportunities. Over small time intervals, by approximate self-financing, the proceeds to accomplish this strategy are obtained from the money market by borrowing (lending) at the risk-free rate. Also, for each portfolio, we present vega contours in panel (b) (even numbered figures). These indicate the effects on the portfolio of exogenous (i.e., unanticipated) changes in volatility. Of course, in a constant-volatility world, this panel adds no useful information; in the actual world, it is an important component of a hedging policy. Previous to taking a position in an options contract, we can use panel (b) to quantitatively indicate which replicating portfolios are relatively volatility resistant.

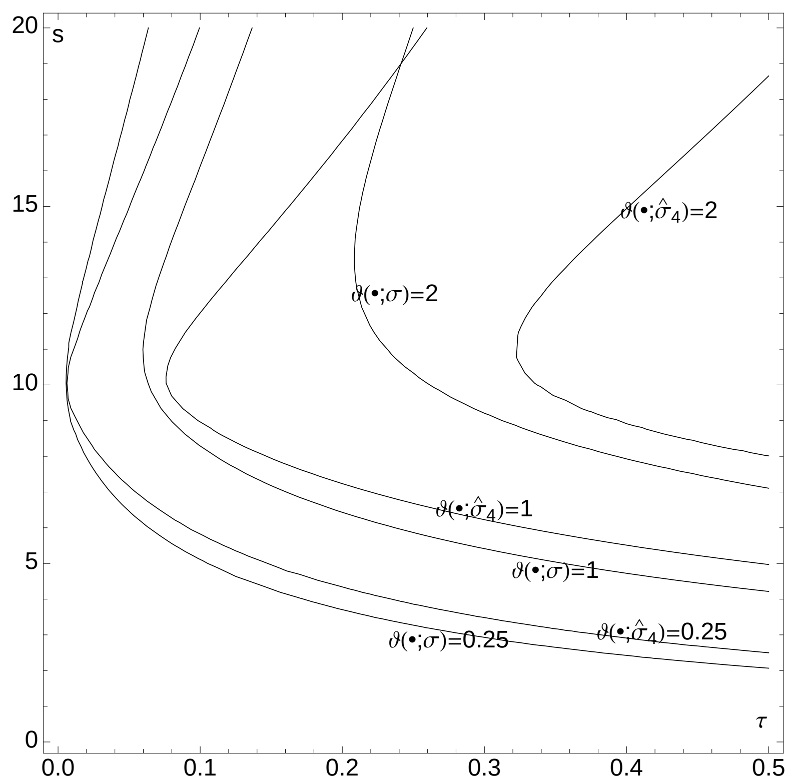
Figures 10–11 present a number of potential delta hedges on a vanilla call placed on a non-dividend paying stock. In particular, illustrated for this portfolio is the difference between pre- and post-estimation constant delta contours. Figure 11, for the instantiating parameters, indicates constant level sets having the property $\{(\tau, s) \in \mathbb{R}^2_+ : \delta(\boldsymbol{v}_t; \star) = \delta_0 \in (0,1)\}$, $\star = \sigma$ or $\widehat{\sigma}_4$. After shorting a call with a stock price that is severely depressed (the blue GBM path), the difference between pre- or post-estimation delta is small. In contrast, for a stock which rapidly appreciates after the call is sold (the green path), the question of "what is the correct delta" for hedging is consequential. As a general proposition, the difference in pre- or post-estimation delta is least crucial for τ near zero, i.e., close to expiry. With τ large, there are substantial differences in pre- and post-estimation δ. Assuming the possibility for changes in σ, similar remarks apply to vega hedges in Figure 10(b). With the underlying price high relative to ATM, if an exogenous σ-shift occurs, there is substantial potential for change in the value of the portfolio. It is obvious that the capacity and magnitude for incorrect hedging is path dependent and potentially large.

In Figure 12(a) we observe the portfolio expiry payoff and its current $t = 0$ value for a long straddle, i.e., the simultaneous purchase of a call and a put, with the same strike and expiry. Figure 13 illustrates a representative set of delta contours associated with the instantiating parameters of this straddle. The portfolio delta satisfies $\{(\tau, s) \in \mathbb{R}^2_+ : 2\delta(\boldsymbol{v}_t; \star) - 1 = \delta_0 \in (-1,1)\}$ for $\star = \sigma$ or $\widehat{\sigma}_4$. After the straddle is shorted, for the purpose of delta hedging, a move to high underlying prices ($S(t;\sigma) > 11$) indicates "some reason for concern" with respect to the accurate estimation of the volatility parameter. With little remaining time, there are small differences in delta hedges. Panel 12(b) provides a view on the vega contours, i.e., the multipliers for $\mathrm{d}\sigma$. These multipliers (points on a contour) are large and allow for substantial mis-hedging when the underlying price path unexpectedly breaks out to the up-side and remains high.

Figure 15 shows representative delta contours ($\sigma = 1$, $s = 10$, $n = 4$, $r = 0.05$, and $T = 1/2$) for a bull spread — this portfolio (see also, Figure 14) mixes a long call and a



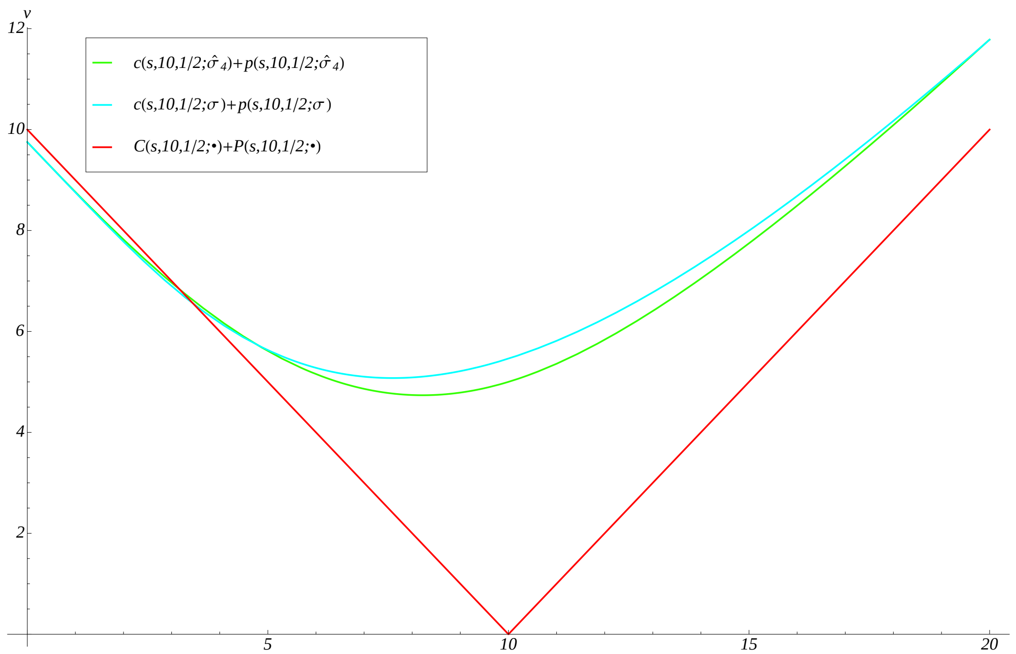
(a) Call Portfolio



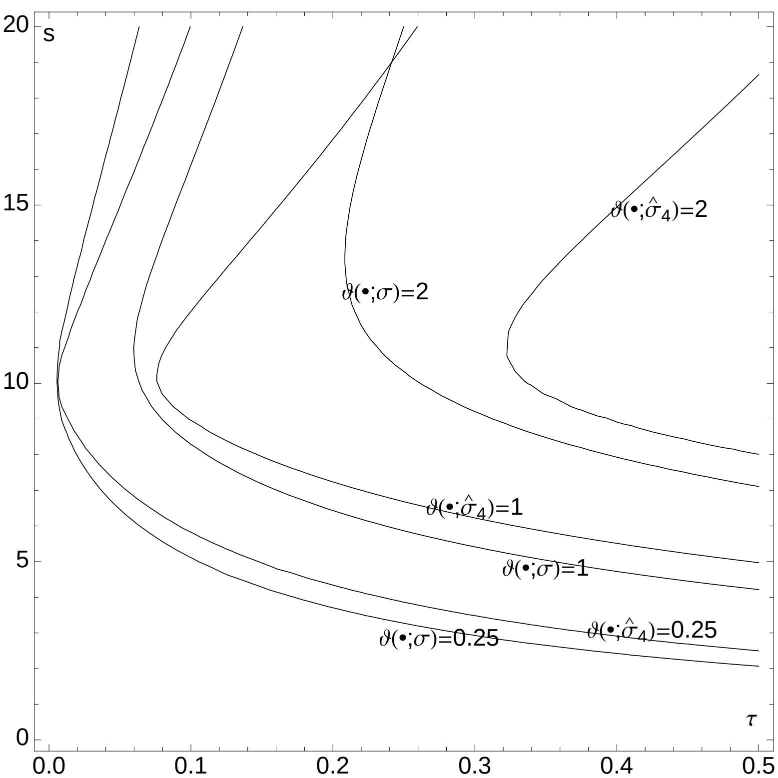
(b) Vega Contours Call

Figure 10: A single call option: (a) six months pre-expiry and expiry value profile; (b) potential on profits of volatility shifts.

Figure 11: A single call option: delta contours.



(a) Long Straddle Portfolio



(b) Vega Contours Long Straddle

Figure 12: Long straddle: (a) six months pre-expiry and expiry value profile; (b) potential on profits of volatility shifts.

Figure 13: A long straddle: delta contours.

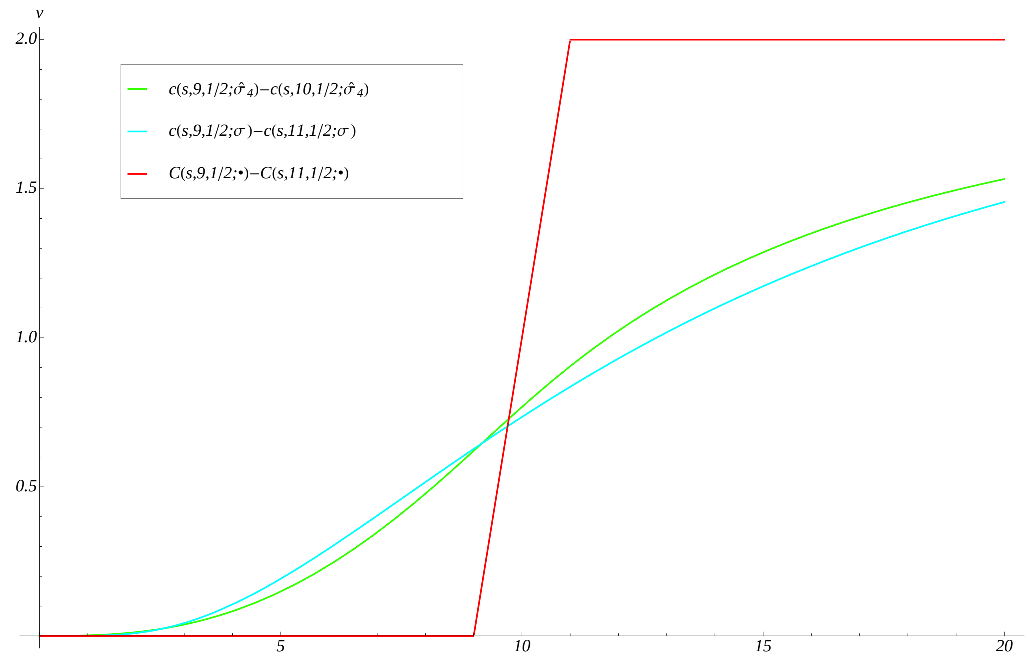
short call with the same expiry date, but with the former having a lower strike price than the latter. For our example, the long call is struck at $k = 9$, whereas the short call has a strike of $k = 11$. The bull spread exhibits complicated behavior over time, and consequently the potential for severe mis-hedging exists. A high underlying price path, for someone who has sold the bull spread, seems to produce the greatest chance for instituting incorrect delta hedging. Panel 14(b) reflects relatively low vega risk throughout a large price range. In fact, there exist prices for which unanticipated increases in σ provide a positive return to the portfolio (see the vega contour having value -0.1).

The final example in this set is illustrated in Figures 16–17 and refers to a portfolio of options called a "butterfly spread" — short two calls with strike $k = 10$, accompanied by a long call with strike $k = 6$, and another long call with strike $k = 14$. The remaining parameters set at $\sigma = 1$, $s = 10$, $n = 4$, $r = 0.05$, and $T = 1/2$. Once again, for someone who is short the butterfly spread, the delta contours in Figure 17 exhibit the possibility of misjudgment and a consequent monetary penalty. The vega effect in Panel 16(b) is generally negative, though in magnitude not as large as the vega effect for the long straddle.

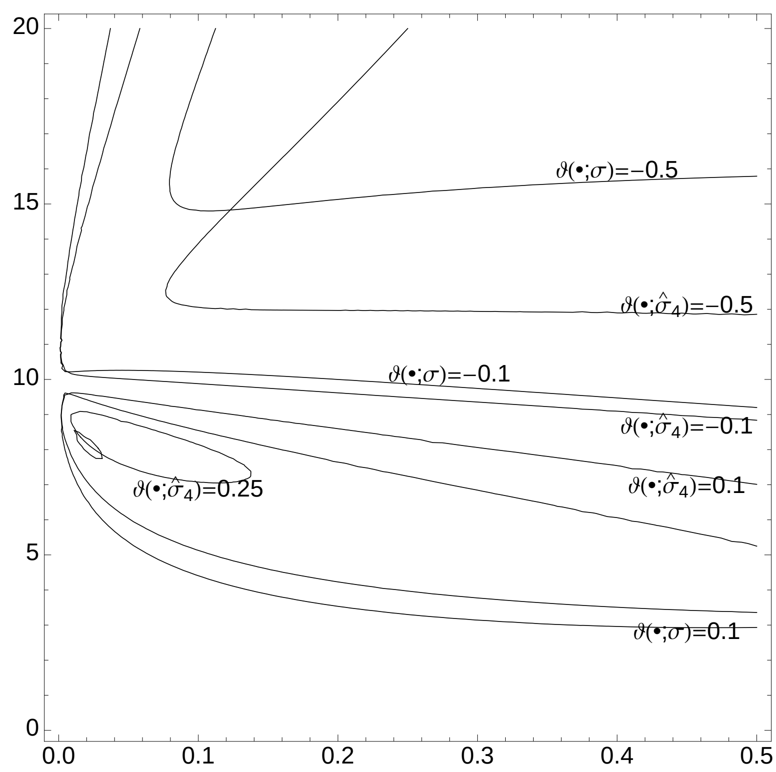
In fact, the potential hedging error is not quite as large as alluded to in the previous examples. The reason, just as in an earlier example, is that as time proceeds we acquire information about the stock price, hence updating the estimator $\widehat{\sigma}_3$ of σ to $\widehat{\sigma}_4, \widehat{\sigma}_5, \ldots$. If volatility does not change, then over time the contours under the post-estimation case approach the contours for the known-σ case. Only the initial re-hedges with a large τ before expiry will show a substantial difference. However, the issue is yet more complicated since σ does change unpredictably over time in the real world. Hence, when "appropriate," we re-estimate σ and re-start the hedging process. The estimation risk of the portfolios, subject to their instantiations, is summarized in Table 9.

Table 9: Overall Estimation Risk

	vanilla call	call straddle	bull spread	butterfly
delta	moderate	moderate	high	high
vega	high	high	low	low



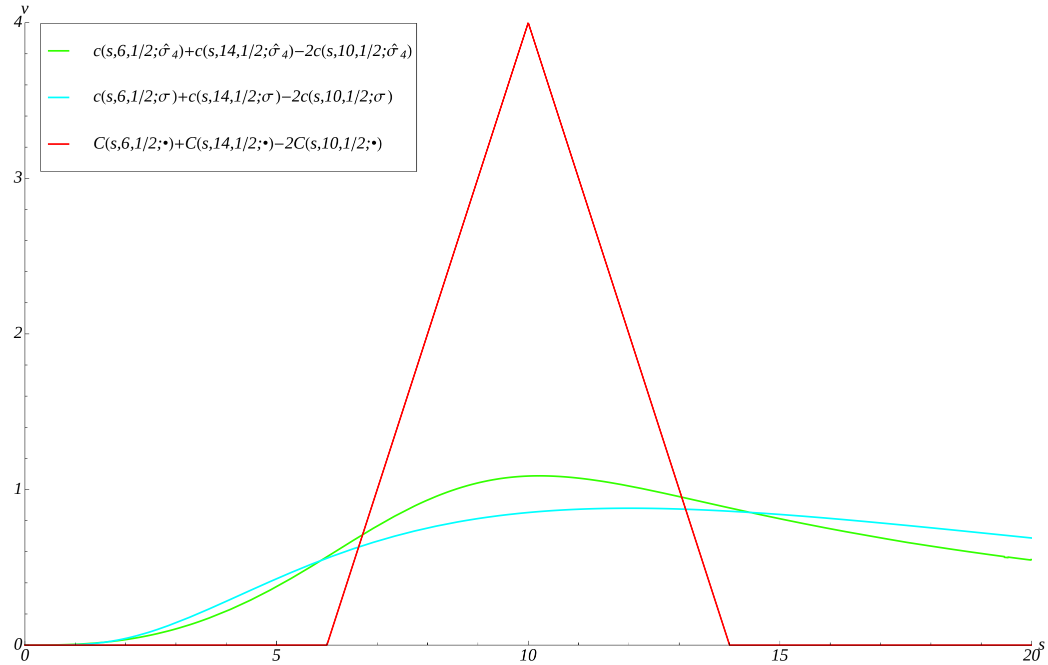
(a) Bull Spread Portfolio



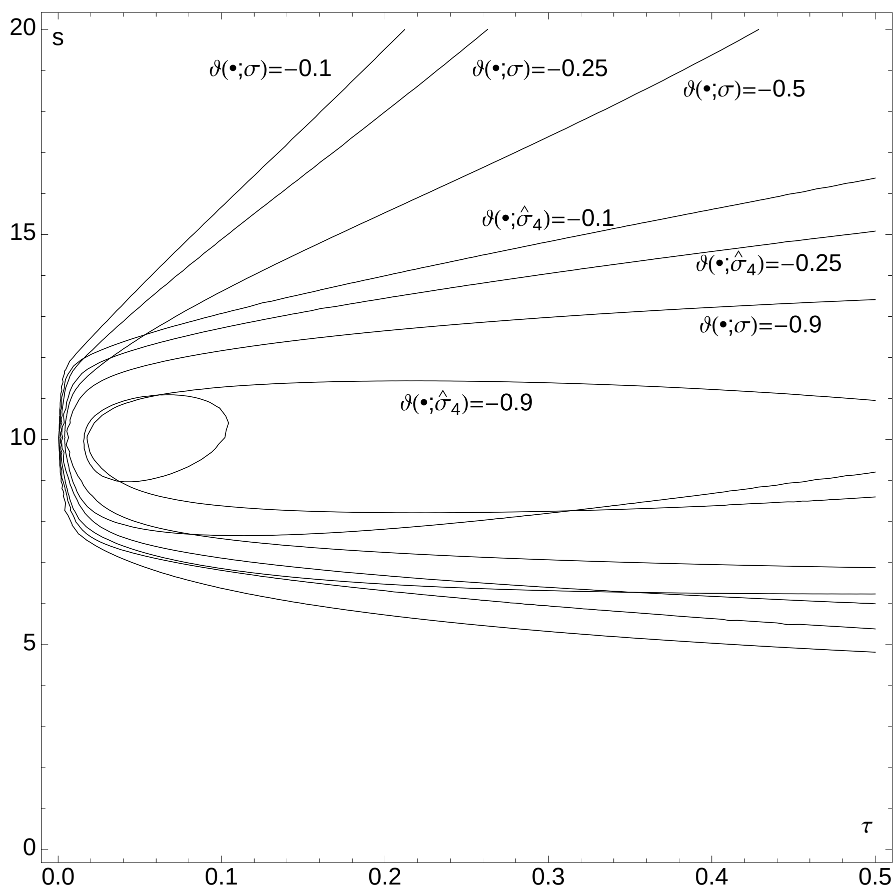
(b) Vega Contours Bull Spread

Figure 14: A bull spread: (a) six months pre-expiry and expiry value profile; (b) potential on profits of volatility shifts.

Figure 15: A bull spread: delta contours.



(a) Butterfly Spread Portfolio



(b) Vega Contours Butterfly

Figure 16: A bull spread: (a) six months pre-expiry and expiry value profile; (b) potential on profits of volatility shifts.

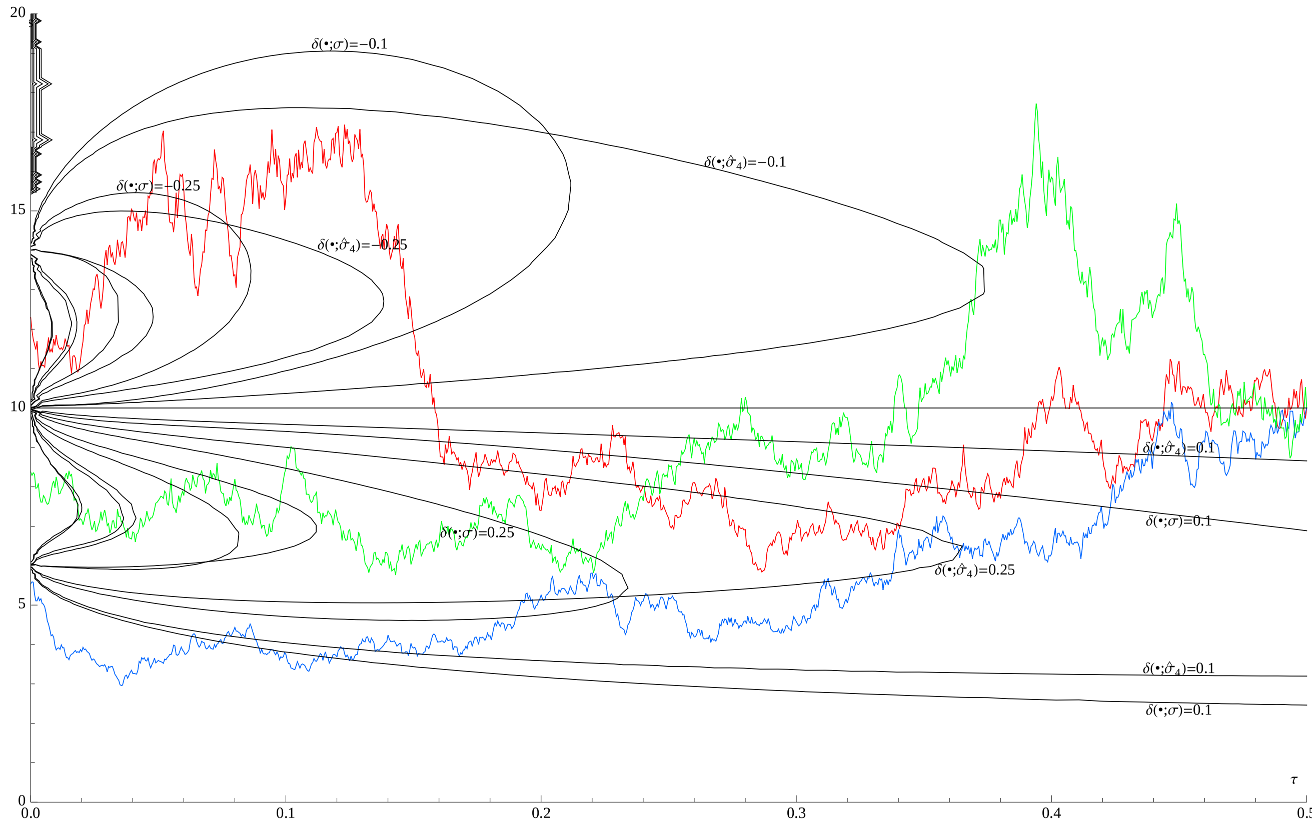


Figure 17: A butterfly spread: delta contours.

3.6 Conclusions

3.6.1 Remarks on Hedging of Portfolios

Recall that there are three possible deltas that can be utilized during the dynamic hedging process: $\delta(\boldsymbol{v}_t;\sigma)$, $\delta(\boldsymbol{v}_t;\widehat{\sigma}_n)$, and $\widehat{\delta}(t) \equiv \Phi(\widehat{z}_+)$. Clearly, this statement holds with respect to any of the option sensitivities, e.g., there are three gamma types, three vegas, etc. In any case, replicating the European option via "delta" hedging will differ when $\widehat{\sigma}_n$ is an estimation-dependent random process. As information via the filtration $(\mathcal{F}_t,\, t \geq 0)$ is acquired, updating the estimate of the volatility will occur, e.g., $\widehat{\sigma}_n$ to $\widehat{\sigma}_{n+1}, \widehat{\sigma}_{n+2}, \ldots,$ for any $n \geq 2$. This will produce a convergence of the δ hedge to that of the ideal BSM δ hedge. However, the BSM value of the option will not be replicated.

Our previous discussion is intended to be illustrative of estimation risk — devoid of unnecessary complexity. Hence, we picked the one critical parameter σ attached with the case of constant-coefficients BSM contingent claim valuation. If one prefers additional complexity, our results can be applied to a joint c.d.f. determining the laws of motion attached to multiple equities. We can also take volatility to be a process parameterized by several constants or further deeper-level underlying processes. All these, in turn, will have estimators attached to them with obvious repercussions for derivative valuation.

A point of contention can arise when simulating a single GBM path or emulating from a historical security path. In particular, given initial data that instantiate the option and underlying price process, it is possible to obtain an atypical random draw. When such a price path arises in a one-on simulation, the implied volatility and realized volatility may be of different orders of magnitude. The caveat for such a one-on case is that the modeling must incorporate a methodology for constructing, e.g., Abu-Mostafa [1], a *typical* path upon which hedging is to take place. For instance, when testing a trading policy by emulating a real-world price path, we can use the observed *typical* returns to construct a confidence or prediction interval for the volatility parameter σ. Given the chosen confidence interval, it is then possible to enforce a trading rule (policy) based on observation of the value of implied volatility relative to realized volatility and the constructed confidence interval.

3.6.2 Summing-up

Obvious steps for extending our analysis call for placing the various hedging policies, inclusive of transaction costs, within the context of an optimization framework. Several possibilities come to mind. Due to the updating of the volatility estimate, the portfolio can be placed in the context of a stochastic program with recourse [51]. A portfolio should be subject to various fixed and unit costs. One can then proceed to calculate the optimal "delta contour" policy. Of interest is the testing of rules on realized versus implied volatility arbitrage. The rules may be in the form of confidence intervals for σ, and there are many other possibilities open for examination. Also, adding a statistical component to the hedging policy that takes account of correlated hedging error can be of use in developing efficient hedging policies. Finally, incorporating change point analysis as a bridge between estimation error and stochastic volatility models is worth exploring. In this context, introducing an estimated function of volatility and other pertinent financial variables is an extension that can be pursued, i.e., in this case, we incorporate an estimator of *all* variables into the model.

3.7 Appendix

At this point we illustrate, by way of an example, another possible dynamic delta hedging policy. Delta level sets or contours are a useful visualization tool for this purpose and are defined as the set of points $(\tau, s) \in \mathbb{R}^2$ satisfying for some chosen $\delta_0 \in \mathbb{R}$, $\delta(\boldsymbol{v}_0; \sigma) = \delta_0$. By differentiability of the option sensitivity delta and the implicit function theorem, we have for given δ_0, a unique (τ_0, s_0), satisfying $\frac{\mathrm{d}s}{\mathrm{d}\tau} = \frac{\partial \delta(\boldsymbol{v}_0;\sigma)}{\partial t} / \frac{\partial \delta(\boldsymbol{v}_0;\sigma)}{\partial s}$. This value indicates the necessary local change in equity price s needed, on account of a time-bleed of τ, so that no additional adjustment of the hedge is required via the position in the underlying. In a world of known σ, rebalancing the portfolio at each delta contour amounts to the continuous replication of the option. If markets are frictionless, i.e., entry and exit costs are zero, the accumulated and appropriately discounted stream of incremental profits — possibly negative at times — will equal the current fair price of the option. If there are costs to transacting, the market will lack the property of completeness [55], i.e., roughly speaking, there are more sources of randomness then ways to hedge them. Leland [37] proposes a volatility mark-up in such situations. The issue of market completeness also comes up in the stochastic volatility literature and our introduction of "estimation" risk is very much akin to that. In a future paper using the tools of estimation and simulation, we compare and test the costs and properties of optimally hedge-replicating contingent claim portfolios subject to market incompleteness.

Our contour map setup is flexible for a variety of hedging policies. Consider the policy of discrete re-hedge monitoring on an a priori chosen set of delta contours. As we have noted previously, when hedging is discrete, self-financing fails in the strict sense, i.e., depending on the chosen discrete hedging policy, there is typically a modest external sourcing of funds to the portfolio. Starting from the initial value s_0, whenever the underlying stock price crosses one of the *chosen* contours, we re-balance by buying or selling δ units of the underlying and appropriately adjusting the money market account. Since GBM at any given contour and time interval is by nature very erratic, re-crossing the same contour after just having crossed it will not signal another re-hedge. Clearly, if trading costs are explicitly considered, the choice of the set of delta contours matters — especially when trading against implied volatility. Surely, the chosen set of contours depends on the risk tolerance of the hedging institution or individual. Explicitly including the cost of transacting, a policy of too many hedging contours will overwhelm any accrued profits, and too few contours will prevent a reasonable hedge from being implemented in a timely way. Thus, there is a "risk-reward" tradeoff for each possible ensemble of contours on which a hedging policy is executed.

Figure 18 depicts such a "contour" hedging policy consisting of nine a priori chosen δ level sets — indicators of when to activate a trade — with δ values 0.1, 0.2, …, 0.9. The remaining contract time τ is indicated on the horizontal axis and the stock price is measured on the vertical axis. The stock price time series runs from right to left, with initial value $s_0 = 10$. We assume a strike of $k = 10$, a risk-free rate of $r = 0.05$, and a known $\sigma = 1$. The long call option $C(\boldsymbol{v}_0; \sigma)$ has an expiry in six months, i.e., $T = 1/2$. Though the sample price path is a "continuous" depiction of GBM, monitoring of the call option is assumed to occur on a set of discrete time points $\{t_1, t_2, \ldots, t_m\}$. We see from the diagram that the price path visits contours 0.3 through 0.9 on the time-to-go τ interval [0.5, 0.12]. Over this interval there are fifteen hedges $x_1, x_2, \ldots, x_{15}$ activated with five of them occurring rapidly over $\tau \in [0.145, 0.12]$. On τ interval $(0.12, 0]$, only two hedges x_{16} and x_{17} are implemented. Essentially, the policy activates a hedge provided the option is relatively (depending on τ) close to $s_T = 10$, i.e., at-the-money (ATM). For the given GBM path sample, no re-hedging occurs close to $\tau = 0$, since the option is, in a relative sense, distant from ATM. Note that if the GBM path remains sufficiently close to $(\tau, S(t;\sigma)) = (0, 10)$, hedging will be successively more frequent, since the option value is always closer yet to ATM.

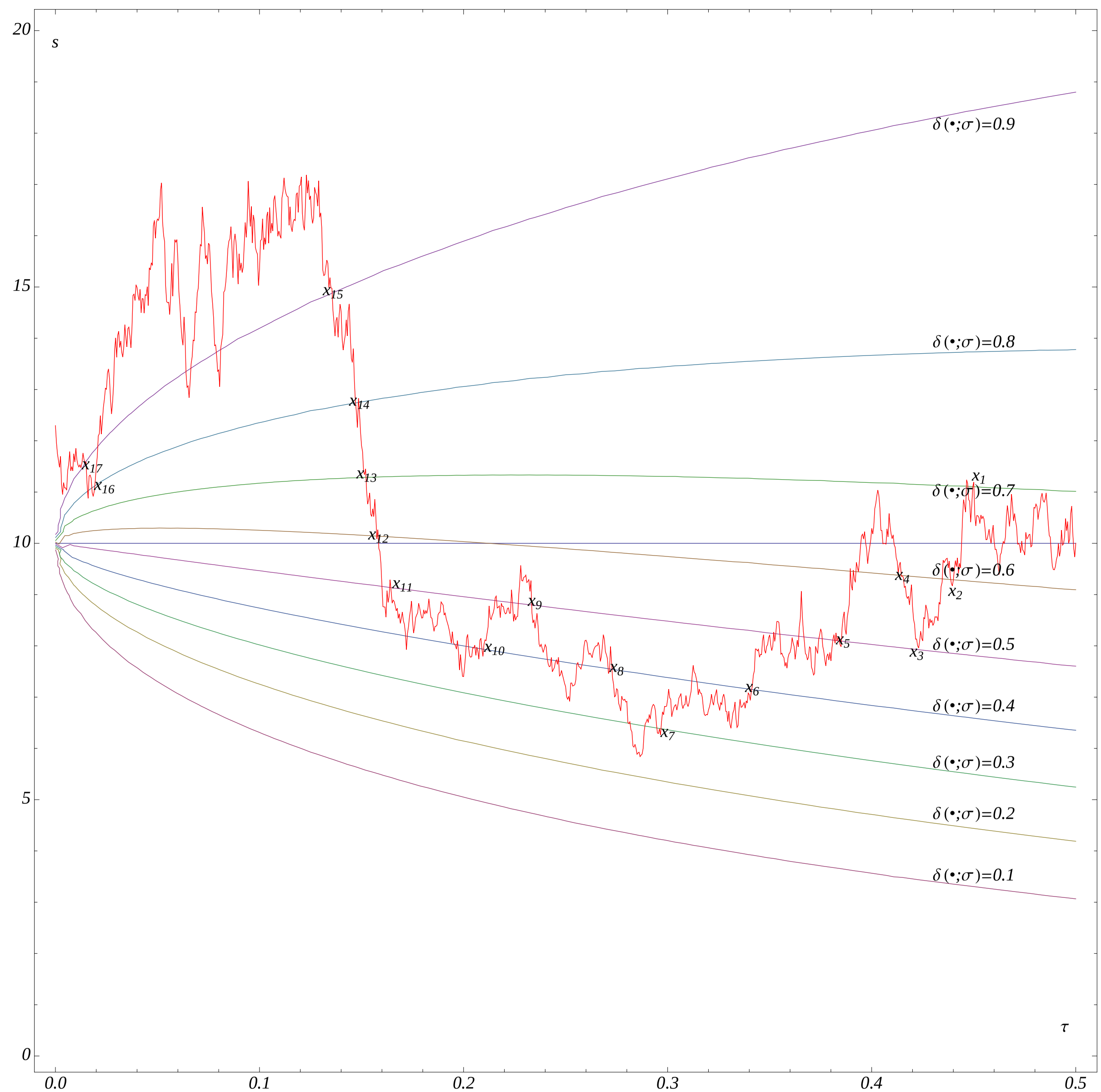


Figure 18: Delta hedging a vanilla call

CHAPTER IV

CONCLUSIONS AND FUTURE RESEARCH

It is surprising that the inclusion of estimation risk in the valuation and hedging of options is a lagging component in the theory of contingent claims. After all, the more-or-less simultaneous papers of Black and Scholes [9] and Merton [41] that initiated the explosive growth of derivative products and organized markets are now over 30 years old. We surmise that the use of implied volatility as a convenient tool has been, until recently, one of the main propagators of the status quo in this area. The use of implied volatility, as a measure of actual volatility or as a method of calibration, leads one to naturally forego the necessary work needed to better understand the process of volatility creation. At the present time, our work can be considered as being at the intersection of two active research areas. The first deals with stochastic volatility models and the second is the exploration of so-called realized volatility.

One of Carr's FAQs (no. 2) [16] is "Why don't the statistical probabilities matter in the binomial model?" As statistical probabilities generally do not matter in classical BSM, we have all the more reason not to use them, and therefore we observe the great stress in the past 30 years on implied σ. Also, the general make-up of many so-called "Quant" departments in industry, due to the newness of the field, contains a large proportion of numerical analysts. Myopia, attempts to satisfy regulators, or a lack of technical ability by management unfortunately place a premium on model "fit" as *the* indicator of risk. The consequence is that short-term fit is stressed over long-term model comprehension. Quants know how to fit "anything" given a set of data to some (any) model, e.g., use Newton-Raphson to obtain implied volatility; then use sophisticated numerical algorithms to fit the parameters to a more-comprehensive in-house model. And this is the problem — a lack of vision reinforced by the pressure to generate immediate results.

4.1 Possible Extensions

Finally we outline some areas where additional effort, along the lines we espouse, can lead to a better comprehension of model and estimation risk.

4.1.1 Basic Extensions: Multivariate Case

We assume two equities $(S_1(t;\Sigma), S_2(t;\Sigma))$ — easily generalizable to many — having returns governed by the laws of motion

$$\begin{aligned} R_1(t;\Sigma) &\equiv \ell\mathrm{n}\Big(\frac{S_1(t+\Delta;\Sigma)}{S_2(t;\Sigma)}\Big) = (\mu_1 - \tfrac{\sigma_1^2}{2})\Delta + \sigma_1 \mathrm{d}\mathcal{W}_1(t), &(46)\\ R_2(t;\Sigma) &\equiv \ell\mathrm{n}\Big(\frac{S_2(t+\Delta;\Sigma)}{S_2(t;\Sigma)}\Big) = (\mu_2 - \tfrac{\sigma_2^2}{2})\Delta + \sigma_2 \mathrm{d}\mathcal{W}_2(t), &(47)\end{aligned}$$

where

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \tag{48}$$

specifies the correlation structure between the two process returns and consequently the two equities. Formally, the dependence structure between the two Brownian increments is $\mathrm{E}[\mathrm{d}\mathcal{W}_1(t)\mathrm{d}\mathcal{W}_2(t)] = \rho \mathrm{d}t$. This is the crucial component that induces the second moment structure reflecting the degree of dependence in the portfolio through $\mathrm{Var}[R_1(t;\Sigma)] = \sigma_1^2\mathrm{d}t$, $\mathrm{Var}[R_2(t;\Sigma)] = \sigma_2^2\mathrm{d}t$, and $\mathrm{Cov}(R_1(t;\Sigma), R_2(t;\Sigma)) = \rho\sigma_1\sigma_2\mathrm{d}t$. So we need to estimate three parameters in all: σ_1, σ_2, and ρ.

In order to use the Cholesky transformation $\mathcal{C}$, which is the analogue of the square root of a positive number, but applies to the case of a positive definite matrix, we set $\Sigma = \mathcal{C}\mathcal{C}^T$. By equating the coefficients of Σ and $\mathcal{C}\mathcal{C}^T$ it follows that

$$\mathcal{C} = \begin{pmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sigma_2\sqrt{1-\rho^2} \end{pmatrix}. \tag{49}$$

Thereafter, consulting the Wishart c.d.f. [28] it may be possible to extend the results of Chapters 2 and 3 to the multivariate case.

4.1.2 Bayesian Case

Another possible extension of the model consists of placing a prior c.d.f. on σ. Clearly, the Bayesian approach contradicts the Rational Expectations Hypothesis (REH), though given a stationary economy, in the limit both converge to the same equilibrium. In line with the Bayesian approach and the gestalt of information generation, one can treat n as a compound Poisson process. This fits a stochastic volatility model where the agent/trader receives at rate λ a random number of arrivals of "information" indicating what the current underlying price is. Foundation work in this market micro-struture area has been done by Diebold [19] and Garman [22].

4.1.3 Comprehensive Risk Analysis Under Alternative Stochastic Processes

We used a well-known process governed by BM. There are other stochastic processes that distribute valuation and hedging risk, e.g., more-general Lévy processes [43], square-root diffusions, constant elasticity of variance processes, and many other candidate processes [23]. Clearly, the Ho–Lee model referenced in [23] is an easy model to subject to our sort of risk inclusion as it is driven by strictly ordinary BM.

4.1.4 Fixed Point Analysis and the Connection Between a Command Economy and a Competitive Equilibrium

The final area of study we cite is directly related to the economics of information and the REH. It would be desirable to characterize the equilibrium behavior of an economy that is updating its information concerning the underlying processes. Essentially, we have proposed a sequence of economies indexed by n tending to the BSM economy as n grows large. One, among many, questions of interest [56] deals with the equilibrium n observed in markets exhibiting various degrees of liquidity.

4.2 Appendix: R Functions

This appendix reproduces the main R functions utilized in the thesis.

4.2.1 Brownian Motion

```
BM <- function(B0, mu, sigma, t0, T, N, paths)
{ # BM(B0=0,mu=0.05,sigma=1,t0=0,T=1,N=5,paths=8)
  # begin BM generates Brownian motion paths of equally spaced realizations.
  # B0 = initial value of Brownian motion
  # mu = drift (annualized)
  # sigma = volatility (annualized)
```

```
  # t0 = initial time
  # T = terminal time for BM
  # N = samples on [t0,T]
  # paths = number of BM sample paths

  # set.seed(123321, kind = NULL) # set random number seed for path generation

if (T <= t0)
        stop("impossible times")
dt <- (T-t0)/N
t <- seq(t0,T,length = N+1)
Bdrift <- rbind(B0,mu*dt*matrix(rep(1, N*paths),N,paths))
Brandom <- rbind(0,matrix(rnorm(N*paths),N,paths))*sigma*sqrt(dt)
B <- ts(apply(matrix(Bdrift+Brandom,N+1,paths),2,cumsum),start=t0,frequency=N)

return(B)

} # end BM
```

4.2.2 Geometric Brownian Motion

```
GBM <- function(s, mu, sigma, t0, T, N, paths)
{ # GBM(s=10,mu=0.05,sigma=1,t0=0,T=1/6,N=176,paths=10^5)
  # begin GBM generates Brownian motion paths of equally spaced realizations.
  # s = initial value of geometric Brownian motion
  # mu = drift (annualized)
  # sigma = volatility (annualized)
  # t0 = initial time
  # T = terminal time for GBM
  # N = samples on [t0,T]
  # paths = number of BM sample paths

  # set.seed(1233216, kind = NULL) # set random number seed for path generation

dt <- (T-t0)/N
Drift <- apply((mu-sigma^2/2)*dt*matrix(rep(1,N*paths),N,paths),2,cumsum)
B <- BM(B0=0,mu=0,sigma,t0,T,N,paths) # standard BM sigma volatility
Noize <- B[2:nrow(B),]
```

```
G <- ts(rbind(s,s*exp(Drift+Noize)),start =t0,frequency=N)

return(G)

} # end GBM
```

Note that log-returns can be obtained by the R command:

```
gbmR <- log(gbm[2:NROW(gbm),]/gbm[1:(NROW(gbm)-1),])
```

4.2.3 Post-Estimation GBM

```
GBM_PostEst <- function(s, mu, sigma, t0, T, df, N, paths)
{ # GBM_PostEst(s=10,mu=0.05,sigma=1,t0=0,T=1/6,df=3,N=176,paths=10^5)
  # begin GBM_PostEst generates exact post-estimation Brownian
  # motion paths with equally spaced realizations.
  # s = initial value of geometric Brownian motion
  # mu = drift (annualized)
  # sigma = volatility (annualized)
  # df = degrees of freedom (df=n-1)
  # t0 = initial time
  # T = terminal time for GBM
  # N = samples on [t0,T]
  # paths = number of BM sample paths

  # set.seed(321321, kind = NULL) # set random number seed for path generation
    Omega <- matrix(sigma^2*rchisq(paths,df,ncp=0)/df,1,paths) # random path volatility

 # set.seed(1233216, kind = NULL) # path generation -- invoke CRN with GBM
dt <- (T-t0)/N
Drift <- apply(matrix(rep((mu-Omega/2)*dt,N),N,paths,byrow=TRUE),2,cumsum)
Noize <- apply(matrix(rep(sqrt(Omega*dt),N),N,paths,byrow=TRUE)
                   *matrix(rnorm(N*paths),N,paths),2,cumsum)
Gpostest <- ts(rbind(s,s*exp(Drift+Noize)),start=t0,frequency=N)

DataOut <- list(Gpostest = Gpostest, Omega = Omega)
names(DataOut)[[1]] <- "Gpostest"  # "Post_Estimation GBM"
names(DataOut)[[2]] <- "Omega"  # "volatility estimator"
return(DataOut)
```

```
} # end GBM_PostEst
```

Log-returns (logR) can be obtained by using the R commands:

```
g <- GBM_PostEst(s=1,mu=0.05,sigma=1,t0=0,T=1,df=3,N=5,paths=8)
gp <- g[1]
logR <- log(gp[2:NROW(gbm),]/gp[1:(NROW(gbm)-1),])
```

4.2.4 Arithmetic-Geometric Average

```
ArithGeomAvg <- function(tS, m)
{ # ArithGeomAvg(tS,m=44)
  # begin ArithGeomAvg - corresponding arithmetic-geometric average along a path
  # tS = underlying time series with nrow = m and ncol = paths
  # m = number of equally spaced points sampled on a path with support [t0,T]
  # t0 = initial time for tS
  # T = terminal time for tS

N <- NROW(tS)
v <- seq(0, N, m)

if(v[length(v)]/m != (N-1)/m)
     stop("!!points m do not divide into time series!!")

tS <- tS[2:N, ]
SubSettS <- tS[v[2:length(v)], ]
tSarith <- apply(SubSettS,2,mean)
tSgeom <- (apply(SubSettS,2,prod))^(1/NROW(SubSettS))

DataOut <- list(tSarith=tSarith,tSgeom =tSgeom)
names(DataOut)[[1]] <- "tSarith"  # "arithmetic average"
names(DataOut)[[2]] <- "tSgeom"  # "geometric average"
return(DataOut)

} # end ArithGeomAvg
```

We present the formulae used in pre- and post-estimation valuation.

4.2.5 BSM Call

```
BSMCall <- function(s,k,mu,T,sigma)
{ #BSMCall(s=10,k=11,mu=0.05,T=1,sigma=1)
 # begin BSM - Black-Scholes-Merton formula
 # s = initial value
 # k = strike
 # mu = risk-free interest rate
 # T = terminal time for G
 # sigma = volatility (annualized)

z1 <- (log(s/k)+(mu+0.5*sigma^2)*T)/sqrt(sigma^2*T)
z2 <- (log(s/k)+(mu-0.5*sigma^2)*T)/sqrt(sigma^2*T)
c <- s*pnorm(z1) - k*exp(-mu*T)*pnorm(z2)

return(c)
} # end BSM
```

4.2.6 BSM Geometric Average Call — Continuous Monitoring

```
BSMGeoAvgCts <- function(s, k, mu, T, sigma)
{ #BSMGeoAvgCts(s=10,k=10,mu=0.05,T=1/3,sigma=1)
 # begin BSMGeoAvgCts - Black-Scholes-Merton formula for continuous geometric average
 # s = initial value
 # k = strike
 # mu = risk-free interest rate
 # T = terminal time
 # sigma = volatility (annualized)

z1 <- (log(s/k)+(mu+sigma^2/6)*T/2)/(sigma*sqrt(T/3))
z2 <- (log(s/k)+(mu-sigma^2/2)*T/2)/(sigma*sqrt(T/3))
c <- s*exp(-mu*T/2 - sigma^2*T/12) * pnorm(z1) - k*exp(-mu*T) * pnorm(z2)

return(c)

} # end BSMGeoAvgCts
```

4.2.7 BSM Simulated Call

```
BSMsimCall <- function(S,k,mu,T)
```

```
{ # begin BSMsimCall(S,k=10,mu=0.05,T=1/6)
 # ** Black-Scholes-Merton call simulation **
 # S = ensemble of terminal values
 # k = strike
 # mu = risk-free interest rate
 # T = expiry time

callvector <- exp(-mu*T)*pmax(S - k,0) # call realizations
call <- sum(callvector)/length(S) # call valuation estimate (discounted)
stderror <- sqrt(var(callvector))/sqrt(NROW(callvector)) # standard error of estimate

DataOut <- list(callvector = callvector, call = call, stderror = stderror)
names(DataOut)[[1]] <- "callvector"  # "call realizations"
names(DataOut)[[2]] <- "call"  # "call value (discounted expectation)"
names(DataOut)[[3]] <- "stderror" # standard error
return(DataOut)

} # end BSMsimCall
```

4.2.8 Q-Q Plots

The commands used in the Q-Q plots follow. Note the peculiar R commands used in the "text" notation.

```
gbm<-GBM(s=1,mu=0.5,sigma=1,t0=0,T=1,N=1,paths=10^5)
gbm2<-GBM(s=1,mu=0.5,sigma=1,t0=0,T=1,N=1,paths=10^5)  # remember to change seed
gbmp3df<-GBM_PostEst(s=1,mu=0.5,sigma=1,t0=0,T=1,df=3,N=1,paths=10^5)
gbmp9df<-GBM_PostEst(s=1,mu=0.5,sigma=1,t0=0,T=1,df=9,N=1,paths=10^5)
gbmp999df<-GBM_PostEst(s=1,mu=0.5,sigma=1,t0=0,T=1,df=999,N=1,paths=10^5)

par(mfrow=c(2,2))
qqplot(log(gbm),log(gbm2),col="blue",main="(a)",
    xlab="Nor(0,1)",ylab="Nor(0,1)")
abline(a=0,b=1)
qqplot(log(gbmp3df[[1]]),log(gbm2),col="red",main="(b)",
    xlab=expression("log(S(1;"*hat(sigma)[4]*"))"),ylab="Nor(0,1)")
abline(a=0,b=1)
qqplot(log(gbmp9df[[1]]),log(gbm2),col="green",main="(c)",
    xlab=expression("log(S(1;"*hat(sigma)[10]*"))"),ylab="Nor(0,1)")
```

```
abline(a=0,b=1)
qqplot(log(gbmp999df[[1]]),log(gbm2),col="purple",main="(d)",
    xlab=expression("log(S(1;"*hat(sigma)[1000]*"))"),ylab="Nor(0,1)")
abline(a=0,b=1)
```

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